Math 6110

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Course Description: MATH 6110-6120 are the core analysis courses in the mathematics graduate program. The plan for MATH 6110 is to use a combination of the three books by Stein and Shakarchi, devoted to Real, Fourier and Functional Analysis. The main topics to be covered usually vary, but traditionally they include:

1. Abstract measure and integration theory.
2. Differentiation of integrals. Functions of bounded variation. Absolutely continuous functions.
3. Fourier series, Fourier transform.
4. Hilbert spaces, Banach spaces, aspects of spectral theory.
5. Introduction to distribution theory.
6. Basic ergodic theory.

Textbooks: *Real Analysis*, *Fourier Analysis*, and *Functional Analysis* by Stein and Shakarchi.

Lecture 1: Introduction, Motivation (9/3)

Motivation for Lebesgue integration

Let f continuous and 2π-periodic. Consider the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$, where the Fourier coefficients are $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt$.

Define $S_N f(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$

$S_N f \to f$ in $L^2[-\pi,\pi]$, meaning that $\|S_N f - f\|_2 \to 0$ where the L^2 norm is

$$\|g\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^2\, dt\right)^{\frac{1}{2}}$$

Moreover, we have Parseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2$$

in particular, the sequence $(\hat{f}(n))_n$ belongs to l^2.

We would like to say for all a, b,

$$\int_a^b \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}\,dx \stackrel{?}{=} \sum_{n\in\mathbb{Z}} \hat{f}(n)\int_a^b e^{inx}\,dx$$

The right hand side makes sense, since we can integrate

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)\Big[\frac{1}{in}e^{inx}\Big]_a^b$$

so that we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \hat{f}(n)\int_a^b e^{inx} &\le \sum_{n\neq 0}\Big|\hat{f}(n)\Big|\frac{1}{|n|} + (b-a)\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|\,dt \\ &\le (\sum_{m\neq n}\Big|\hat{f}(m)\Big|^2)^{\frac{1}{2}}(\sum_{n\neq 0}\frac{1}{n^2})^{\frac{1}{2}} + (b-a)\|f\|_1 \quad \text{Cauchy-Schwarz} \\ &< \infty\end{aligned}$$

In fact, for all $(a_n)_{n\in\mathbb{Z}} \in l^2$, we would like to consider

$$\int_a^b \sum_{n\in\mathbb{Z}} a_n e^{inx}\,dx$$

but these functions are not in general Riemann-integrable.

To compute the Lebesgue integral, we need a way of measuring the sets of arguments to the function in which the function attains values in certain ranges. It is not obvious how to do this. Thus, measure theory was developed.

Measure theory

First, we work in $\mathbb{R}^d$. We start with some sets that should certainly be measurable: rectangles and cubes.

A **rectangle** is a set of the form $[a_1, b_1] \times \ldots \times [a_d, b_d] \subseteq \mathbb{R}^d$ where $a_i \le b_i$. The volume is $|R| = (b_1 - a_1)\cdots(b_d - a_d)$.

Lemma 1. *Let R be a rectangle that is an* ***almost disjoint*** *union of rectangles $R = \cup_{k=1}^N R_k$, meaing that R_i can only intersect R_j for $i \neq j$ at its boundary. Then*

$$|R| = \sum_k |R_k|$$

Proof. The proof when the subrectangles are aligned in a grid is easy by adding up telescoping sums. If the subrectangles are not aligned in a grid, we can make it into a grid by adding more lines. □

Lemma 2. *If $R, R_1, \ldots, R_N$ are (not necessarily disjoint) rectangles, and if $R \subseteq \bigcup_{k=1}^N R_k$, then $|R| \subseteq \sum_{k=1}^N |R_k|$*

Proof. Clear from a picture. $\square$

Theorem 1. *Every open set $\mathcal{O} \subseteq \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Proof. For any $x \in \mathcal{O}$, there exists a largest open interval I_x such that $x \in I_x \subseteq \mathcal{O}$. In fact, $I_x = (a_x, b_x)$, where $a_x = \inf\{a < x : (a,x) \subseteq \mathcal{O}\}$ and $b_x = \sup\{b > x : (x,b) \subseteq \mathcal{O}\}$. Clearly we have $\bigcup_{x \in \mathcal{O}} = \mathcal{O}$. The collection of such distinct intervals I_x are disjoint. To see this, suppose $I_x \cap I_y \neq \emptyset$. $I_x \cup I_y$ is an open interval, and $x \in I_x \subseteq I_x \cup I_y \subseteq \mathcal{O}$, which forces $I_x \cup I_y = I_x$ by our choice of I_x, so along with the symmetric argument for y, we know that $I_x = I_y$. Any collection of disjoint intervals is countable, since each contains a unique rational number. $\square$

Theorem 2. *Any open $\mathcal{O} \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.*

Lecture 2: Exterior Measure (9/5)

Proof of Theorem 2. Consider the dyadic grid of cubes of scale 1, and let the collection $\mathcal{C}_1$ be the Q such that $Q \subseteq \mathcal{O}$. For a grid of scale $1/2^k$, let $\mathcal{C}_k$ be defined as those cubes Q in the grid that is not contained in $\bigcup_{j<k} \mathcal{C}_j$. We claim that $\mathcal{O} = \bigcup_{Q \in \cup_{k=1}^\infty \mathcal{C}_k} Q$. The $(\supseteq)$ containment is by definition. Now, let $x \in \mathcal{O}$. Then since $\mathcal{O}$ is open, there exists a cube Q_x such that $x \in Q_x \subseteq \mathcal{O}$. Say Q_x is of scale $1/2^s$. Then by construction, it is either part of $\mathcal{C}_s$ or contained in a larger cube in $\mathcal{C}_j$ for $j < s$. This is because it is contained in $\mathcal{O}$: if it is not part of $\mathcal{C}_s$, then it is contained in $\bigcup_{j<k} \mathcal{C}_j$. $\square$

The Exterior Measure

Definition 0.1. Let $E \subseteq \mathbb{R}^d$ be any set. Then we define the **exterior measure**

$$m_*(E) := \inf\Big\{ \sum_{j=1}^\infty |Q_j| \mid E \subseteq \bigcup_{j=1}^\infty Q_j,\ Q_j \text{ are closed cubes} \Big\}$$

Example 0.1. $m_*(\{x\}) = 0$ is clear.

Example 0.2. $m_*(Q) = |Q|$ for a closed cube Q.

First, we know that $m_*(Q) \le |Q|$ by definition because Q covers Q. Consider $Q \subseteq \bigcup_{j=1}^\infty Q_j$ where the Q_j are closed cubes. Then $|Q| \le \sum_{j=1}^\infty |Q_j|$, so of course $|Q| \le \inf \sum_{j=1}^\infty |Q_j|$ where the inf is taken over all collections of closed cubes that contain Q, and hence the inf is equal to $m_*(E)$.

To see that $|Q| \le \sum_{j=1}^\infty$, let $\epsilon > 0$, and choose $S_j \supseteq \mathcal{O}_j$ open cubes with $|S_j| \subseteq |\mathcal{Q}_j|(1+\epsilon)$. Clearly $Q \subseteq \bigcup_{j=1}^\infty S_j$, so by compactness we can choose an open cover $S_1 \cup \ldots \cup S_N$. Then we apply our above Lemma 2 to conclude that

$$|Q| \le |S_1| + \ldots + |S_N| \le (1+\epsilon)|\mathcal{O}_j|$$

Example 0.3. $m_*(Q) = |Q|$ for an open cube Q.

We know that $m_*(Q) \le m_*(\overline{Q}) = \left|\overline{Q}\right| = |Q|$, so that $m_*(Q) \le |Q|$. Now, for any closed cube $Q_0 \subseteq Q$, $|Q_0| = m_*(Q_0) \le m_*(Q)$.

Example 0.4. $m_*(R) = |R|$ for any rectangle R.

As before, $|R| \subseteq m_*(R)$. Consider the grid of all cubes of side length $1/2^k$. $S = \{Q \subseteq R \mid Q \text{ cube}\}$, and $\tilde{S} = \{Q \text{ cube} \mid Q \text{ intersects both } R \text{ and } R^c\}$. We know that

$$\sum_{Q \text{ in grid}} |Q| = \sum_{Q \in S} |Q| + \sum_{Q \in \tilde{S}} |Q| \leq |R| + C \cdot 1/2^k$$

for some constant C. Thus, taking $k \to \infty$, we have $m_*(R) \leq |R|$, since every choice of the grid is a covering of R by closed cubes.

Example 0.5. $m_*(R) = \infty$ is clear.

Proposition 1 (Properties of the exterior measure)**.** The exterior measure has the following properties:

1. Monotonicity: $E_1 \subseteq E_2 \implies m_*(E_1) \leq m_*(E_2)$
2. Countable subadditivity: $E = \cup_{j=1}^{\infty} E_j \implies m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$
3. $E \subseteq \mathbb{R}^d \implies m_*(E) = \inf m_*(\mathcal{O})$ where the inf is taken over open $\mathcal{O}$ such that $E \subseteq \mathcal{O}$.
4. If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.
5. If $E = \bigcup_{j=1}^{\infty} Q_j$ almost disjoint cubes, then $m_*(E) = \sum_{j=1}^{\infty} |Q_j|$

Proof. (1.) Is clear.

(2.) Without loss of generality, $m_*(E_j) < \infty$. For each j, there exists a sequence of closed cubes (Q_j^k) such that $E_j \subseteq \bigcup_{k=1}^{\infty} Q_j^k$ and $\sum_{k=1}^{\infty} \left|Q_j^k\right| \leq m_*(E_j) + \epsilon/2^j$. But then, $E \subseteq \bigcup_j \bigcup_k Q_j^k$ so that

$$m_*(E) \leq \sum_j \sum_k \left|Q_j^k\right| \leq \sum_j (m_*(E_j) + \epsilon/2^k) = \sum_j m_*(E_j) + \epsilon$$

(3.) $m_*(E) \leq \inf m_*(\mathcal{O})$ by monotonicity. For the other direction, let $\epsilon > 0$. Choose closed cubes Q_j such that $E \subseteq \bigcup_{j=1}^{\infty} \mathcal{O}_j$ such that $\sum_{j=1}^{\infty} |\mathcal{O}_j| \leq m_*(E) + \epsilon/2$. Pick open cubes $\mathcal{O}_j \supseteq Q_j$ such that $|\mathcal{O}_j| \leq |Q_j| + \epsilon/2^{j+1}$. Define $\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}_j$, which is open. Then we know that

$$\begin{aligned} m_*(\mathcal{O}) &\leq \sum_{j=1}^{\infty} m_*(\mathcal{O}_j) \text{ countable subadditivity} \\ &= \sum_{j=1}^{\infty} |\mathcal{O}_j| \\ &\leq \sum_{j=1}^{\infty} |Q_j| + \epsilon/2^{j+1} \\ &\leq (\sum_{j=1}^{\infty} |Q_j|) + \epsilon/2 \\ &\leq m_*(E) + \epsilon/2 + \epsilon/2 \end{aligned}$$

(4.) We already have $m_*(E) \leq m_*(E_1) + m_*(E_2)$. For the other direction, take a covering of closed cubes $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ such that $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon$, where all sidelengths Q_j are $< \delta/1000$. Then there is a partition of the cubes into collections A and B such that $E_1 \subseteq \bigcup_{j\in A} Q_j$ and $E_2 \subseteq \bigcup_{j\in B} Q_j$. Finally,

$$m_*(E_1) + m_*(E_2) \leq \sum_{j\in A} |Q_j| + \sum_{j\in B} |Q_j| \leq m*(E) + \epsilon$$

(5.) For any j, let $\tilde{Q}_j \subseteq Q_j$ such that $|Q_j| \leq \left|\tilde{Q}_j\right| + \epsilon/2^j$.

$$\begin{aligned} m_*(E) &\geq m_*\Big(\bigcup_{j=1}^N \tilde{Q}_j\Big) \\ &= \sum_{j=1}^N \left|\tilde{Q}_j\right| \text{ using (4.)} \\ &\geq \sum_{j=1}^N (|Q_j| - \epsilon/2^j) \\ &\geq \sum_{j=1}^N |Q_j| - \epsilon \end{aligned}$$

□

Note that (4) does not quite say that a union of disjoint sets has exterior measure equal to the sum of the exterior measures of the disjoint sets. The disjoint sets have to be separated to apply (4).

Observation 0.1. Even for disjoint sets $E_1 \cap E_2 = \emptyset$ it may be that $m_*(E_1 \cup E_2) \neq m_*(E_1) + m_*(E_2)$.

Lecture 3: Measurability (9/10)

Measurable sets and Lebesgue measure

A set $E \subseteq \mathbb{R}^d$ is said to be **Lebesgue measurable** if for all $\epsilon > 0$, there exists an open set $\mathcal{O}$ such that $E \subseteq \mathcal{O}$ and $m_*(\mathcal{O} \setminus E) \leq \epsilon$. For a Lebesgue measurable set E, we define the measure of E as $m(E) := m_*(E)$. For simplicity we refer to Lebesgue measurable sets as measurable until later.

Observation 0.2. Every open set is measurable.

Observation 0.3. If $E \subseteq \mathbb{R}^d$ has $m_*(E) = 0$, then E is measurable.

For instance, the Cantor set is measurable. Recall that to define it, let $C_0 = [0,1], C_1 = [0,1/3] \cup [2/3,1]$, and so on, and let $C = \cap_{k=0}^{\infty} C_k$ (Insert Cantor set picture).

Proposition 2. A countable union of measurable sets is measurable.

Proof. Say $E = \cup_{j=0}^{\infty} E_j$ where each E_j is measurable. Let $\epsilon > 0$. For all j, there exists an open $\mathcal{O}_j$ such that $E_j \subseteq \mathcal{O}_j$ and $m_*(\mathcal{O}_j \setminus E_j) \leq \frac{\epsilon}{2^j}$. Then the open set $\mathcal{O} = \bigcup_{j=0}^{\infty} \mathcal{O}_j$ has $E \subseteq \mathcal{O}$ and

$\mathcal{O} \setminus E \subseteq \bigcup_{j=0}^{\infty} \mathcal{O}_j \setminus E_j$. Then we have that

$$\begin{aligned} m_*(\mathcal{O} \setminus E) &\le \sum m_*(\mathcal{O}_j \setminus E_j) \\ &\le \sum \frac{\epsilon}{2^j} \\ &\le 2\epsilon \end{aligned}$$

□

Proposition 3. Closed sets are measurable.

Proof. First, let F be closed and bounded and hence compact. Consider an open set $\mathcal{O}$ with $F \subseteq \mathcal{O}$ such that $m_*(\mathcal{O}) \le m_*(E) + \epsilon$. Consider the open set $\mathcal{O} \setminus F$, and split it into almost disjoint cubes $\mathcal{O} \setminus F = \bigcup_j Q_j$. Define $K_n = \bigcup_{j=1}^{n} Q_j$. Then we have $d(F, K_n) > 0$. Hence,

$$\begin{aligned} m_*(\mathcal{O}) &\ge m_*(K_n \cup F) = m_*(K_n) + m_*(F) \\ &= m_*(F) + \sum_j^n m_*(Q_j) \\ \sum_j^n m_*(Q_j) &\le m_*(\mathcal{O}) - m_*(F) \\ &< \epsilon \end{aligned}$$

This holds for any n, so that $\sum_{j=1}^{\infty} m_*(Q_j) \le \epsilon$. Thus, we have that

$$m_*(\mathcal{O} \setminus F) = m_*\Big(\bigcup_{j=1}^{\infty} Q_j\Big) = \sum_{j=1}^{\infty} m_*(Q_j) \le \epsilon$$

For a generic closed F, write $F = \bigcup_{k=1}^{\infty} \Big(F \cap \overline{B}(0,k)\Big)$. Each element of the intersection is compact and hence measurable by above, so that F, which is a countable union of measurable sets, is measurable by Proposition 2. □

Proposition 4. The complement of a measurable set is measurable.

Proof. Let $E \subseteq \mathbb{R}^d$ measurable. For any n, there is an open $\mathcal{O}_n$ such that $m_*(\mathcal{O}_n \setminus E) \le \frac{1}{n}$ and $E \subseteq \mathcal{O}_n$. Consider the measurable set $S = \bigcup_n \mathcal{O}_n^c$. Note that $S \subseteq E^c$, and that $E^c = S \cup (E^c \setminus S)$. We claim that

$$m_*(E^c \setminus S) \le m_*(\mathcal{O}_n \setminus E) \le \frac{1}{n}$$

This is because $E^c \setminus S \subseteq \mathcal{O}_n \setminus E$ for all n. To see this, note that if $x \notin S = \bigcup \mathcal{O}_n^c$, then $x \in (\bigcup_n \mathcal{O}_n^c)^c = \cap_n \mathcal{O}_n$. Thus, E^c is measurable. □

Proposition 5. A countable intersection of measurable sets is measurable.

Proof. $\cap E_j = (\cup E_j)^c$. Apply the previous propositions. □

Theorem 3 (Countable additivity). *If $(E_j)_{j\in\mathbb{N}}$ are disjoint measurable sets and $E = \bigcup_{j=1}^\infty E_j$, then*

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

Proof. Suppose that each E_j is bounded. Of course $m(E) \le \sum_j m(E_j)$ is true due to countable subdadditivity of the outer measure. Since each E_j is measurable, each E_j^c is measurable. Then there exists $\mathcal{O}_j$ such that $E_j^c \subseteq \mathcal{O}_j$ and $m(\mathcal{O}_j \setminus E_j^c) < \epsilon/2^j$. But then $\mathcal{O}_j := F_j \subseteq E_j$ and $m_*(E_j \setminus F_j) \le \epsilon/2^j$ because $E_j \setminus F_j = \mathcal{O}_j \setminus E_j^c$.

For each N, consider $F_1, \dots, F_N$, which are compact and disjoint. Then we have that

$$\begin{aligned} m(E) &\ge m(\bigcup_{j=1}^{N} F_j) \\ &= \sum_{j=1}^{N} m(F_j) \\ &\ge \sum_{j=1}^{N} m(E_j) - \epsilon/2^j \\ &\ge \Big(\sum_{j=1}^{N} m(E_j)\Big) - \epsilon \end{aligned}$$

so that taking limits, $m(E) \ge \sum_{j=1}^\infty m(E_j)$

For the general case, let Q_k be nested cubes with $\bigcup_{k=1}^\infty Q_k = \mathbb{R}^d$. Consider $S_1 = Q_1, S_2 = Q_2 \setminus Q_1, S_3 = Q_3 \setminus Q_2$, and so on. Define $E_{j,k} = E_j \cap S_k$. Then we have $E = \bigcup_j \bigcup_k E_{j,k}$. By the previous result, we have

$$\begin{aligned} m(E) &= \sum_j \sum_k m(E_{j,k}) \\ &= \sum_j m(E_j) \end{aligned}$$

□

Lecture 4: Properties of Lebesgue Measure (9/12)

Note that the exterior measure m_* can 'measure' any subset of $\mathbb{R}^d$, while m can only 'measure' the Lebesgue measurable sets. However, m_* does not satisfy countable additivity in general, while m does.

Corollary 1. Let $(E_k)_k$ be measurable sets in $\mathbb{R}^d$. Then

1. If $E_k \nearrow E$, then $m(E) = \lim_{k\to\infty} m(E_k)$.
2. If $E_k \searrow E$ and $m(E_K) < \infty$ for some K, then $m(k) = \lim_{k\to\infty} m(E_k)$.

Proof. (1) Define $G_1 = E_1$, $G_2 = E_2 \setminus E_1$, and in general $G_k = E_k \setminus E_{k-1}$ for $k > 0$. Then $E = \bigcup_{k=1}^{\infty} G_k$. Note that the G_k are disjoint. Thus,

$$\begin{aligned} m(E) &= \sum_{k=1}^{\infty} m(G_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^{\infty} m(G_k) \\ &= \lim_{n\to\infty} m\Big(\bigcup_{k=1}^{n} G_k\Big) \\ &= \lim_{n\to\infty} m(E_k) \end{aligned}$$

(2) Assume without loss of generality that E_1 has finite measure. Let $G_k = E_k \setminus E_{k+1}$. Then $E_1 = E \cup \bigcup_{k=1}^{\infty} G_k$.

$$\begin{aligned} m(E_1) &= m(E) + \lim_{n\to\infty} \sum_{k=1}^{n-1} m(E_k) - m(E_{k+1}) \\ &= m(E) + m(E_1) - \lim_N m(E_N) \\ m(E) &= \lim_N (E_N) \end{aligned}$$

□

Theorem 4. *Let $E \subseteq \mathbb{R}^d$ measurable. Then for any $\epsilon > 0$,*

1. *There exists an open $\mathcal{O}$ with $E \subseteq \mathcal{O}$ such that $m(\mathcal{O} \setminus E) < \epsilon$.*
2. *There exists a closed F with $F \subseteq E$ such that $m(E \setminus F) < \epsilon$.*
3. *If $m(E) < \infty$, then there exists a compact K such that $K \subseteq E$ and $m(E \setminus K) < \epsilon$.*
4. *If $m(E) < \infty$, then there exists a closed $F = \bigcup_{j=1}^{n} Q_j$ that is a finite union of closed cubes such that $m(E \Delta F) < \epsilon$, where Δ is the symmetric difference.*

Proof. (1) is by definition.

(2) Since E is measurable, E^c is measurable, so there exists an $\mathcal{O}$ such that $E^c \subseteq \mathcal{O}$ and $m(\mathcal{O} \setminus E^c) < \epsilon$. Thus, $\mathcal{O}^c \subseteq E$, and $E \setminus \mathcal{O}^c = \mathcal{O} \setminus E^c$.

(3) Pick a closed $F \subseteq E$ such that $m(E \setminus F) < \epsilon/2$. Consider $K_n = F \cap \overline{B}_n(0)$. Then $K_n \nearrow F$ and $E \setminus K_n \searrow E \setminus F$. Thus, $m(E \setminus K_n) \to m(E \setminus F)$. Choose an n with $m(E \setminus K_n)$ within $\epsilon/2$ to $m(E \setminus F) < \epsilon/2$, and we are done since K_n is compact.

(4) Consider $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ closed cubes such that $\sum_{j=1}^{\infty} |Q_j| \leq m(E) + \epsilon/2$.

$$\begin{aligned}
m(E) + m\Big(\bigcup_j Q_j \setminus E\Big) &= m(\bigcup_j Q_j) \\
&\leq \sum_{j=1}^{\infty} |Q_j| \quad \text{additivity} \\
m\Big(\bigcup_j Q_j \setminus E\Big) &\leq \sum_j |Q_j| - m(E) \\
&\leq \epsilon/2
\end{aligned}$$

Since $m(E) < \infty$, there exists an n large such that $\sum_{j=n+1} |Q|_j < \epsilon/$. Consider $F = \bigcup_{j=1}^{n} Q_j$. Then

$$\begin{aligned}
m(E \Delta F) = m(E \setminus F) + m(F \setminus E) &\leq m\Big(\bigcup_{j=n+1}^{\infty} Q_j\Big) + m\Big(\bigcup_{j=1}^{\infty} Q_j \setminus E\Big) \\
&< \sum_{j=n+1}^{\infty} |Q_j| + \epsilon/2 \ < \ \epsilon
\end{aligned}$$

□

Observation 0.4 (Invariance properties of Lebesgue measure)**.**
For $t \in \mathbb{R}^d$, and $E \subseteq \mathbb{R}^d$ measurable, letting $E + t = \{x + t \mid x \in E\}$, then $m(E) = m(E + t)$.

For $\lambda \in \mathbb{R}_+$, and letting $\lambda E = \{\lambda x \mid x \in E\}$, then $m(\lambda E) = \lambda^d m(E)$.

Note that the σ-algebra of Borel sets is strictly contained in the σ-algebra of Lebesgue measurable sets. A G_δ is a countable intersection of open sets. An F_σ is a countable union of closed sets.

Corollary 2. A subset $E \subseteq \mathbb{R}^d$ is measurable

1. If and only if E differs from a G_δ by a set of measure zero.

2. If and only if E differs from an F_σ by a set of measure zero.

Proof. For any n, there exists an open $\mathcal{O}_n$ such that $\mathcal{O}_n \supseteq E$ and $m(\mathcal{O}_n \setminus E) < 1/n$. Then consider $G = \bigcap_n \mathcal{O}_n$. Then $G \supseteq E$, and $m(G \setminus E) = 0$. □

We now construct an example of a non-measurable set. Consider $\mathbb{R}$ and $[0,1]$. Say $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Consider $[0,1]/\sim = \bigcup_\alpha \mathcal{E}_\alpha$, where the $\mathcal{E}_\alpha$ partition $[0,1]$. Pick one representative x_α for each $\mathcal{E}_\alpha$ (using axiom of choice). The set of all representatives $N = \{x_\alpha\}$ is a non-measurable set.

To see that N is non-measurable, suppose it were, then consider $(N + r_k)_k$, where (r_k) is an enumeration of $\mathbb{Q} \cap [0,1]$. Then $[0,1] \subseteq \bigcup_k (N + r_k) \subseteq [-2,3]$. Then

$$\begin{aligned}
m([0,1]) = 1 &= \sum_k m(N + r_k) \\
&= \sum_k m(N)
\end{aligned}$$

which is not possible, so we conclude that N is non-measurable.

Lecture 5: Measurable Functions (9/17)

Measurable functions

If $E \subseteq \mathbb{R}^d$, denote the characteristic function of E by χ_E or $\mathbb{1}_E$. Note that for $E = [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$, χ_E is not Riemann-integrable, since the upper Riemann sums are always 1, and the lower Riemann sums are always 0, due to the density of the rationals.

Definition 0.2. A **simple function** is one of the form $f = \sum_{k=1}^{N} a_k \chi_{E_k}$, where the E_k are measurable sets of finite measure.

Definition 0.3. Let $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$, in which D is Lebesgue measurable and $f(x)$ can take on $\pm\infty$ but only on a set of measure zero. f is said to be **measurable** if for all $a \in \mathbb{R}$, then set $f^{-1}((-\infty, a)) = \{x \in D \mid f(x) < a\}$ is Lebesgue measurable.

There are many other equivalent ways to define measurability of a function. For instance, we can require $\{f \leq a\}$ to be measurable, since $\{f \leq a\} = \cap_{k=1}^{\infty} \{f < a + 1/k\}$ and $\{f < a\} = \cup_k \{f \leq a - 1/k\}$.

Proposition 6. f is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for all open $\mathcal{O} \subseteq \mathbb{R}$ if and only if $f^{-1}(F)$ is measurable for all closed $F \subseteq \mathbb{R}$.

Proposition 7. If f is continuous on $\mathbb{R}^d$, then f is measurable. Also, if f is measurable and finite valued and φ is continuous, $\varphi \circ f$ is measurable.

Proof.

$$\begin{aligned}
(\varphi \circ f)^{-1}((-\infty, a)) &= \{x \mid \varphi(f(x)) \in (-\infty, a)\} \\
&= \{x \mid f(x) \in \mathcal{O}, \mathcal{O} = \varphi^{-1}((-\infty, a))\} \\
&= f^{-1}(\mathcal{O})
\end{aligned}$$

so we are done since $\mathcal{O}$ is open by continuity of φ. $\square$

Observation 0.5. In general, for continuous φ, $f \circ \varphi$ is not measurable in general.

Proposition 8. Let $(f_n)_n$ be a sequence of measurable functions. Then

$$\sup_n f_n(x) \qquad \inf_n f_n(x) \qquad \limsup_n f_n(x) \qquad \liminf_n f_n(x)$$

are all measurable.

Proof. Note that $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$. Then we can apply this to $\inf f_n(x) = -\sup(-f_n(x))$. Since the $\limsup$ is an inf of a sup, and $\liminf$ is a sup of an inf, we are done. $\square$

Proposition 9. Let f, g measurable real functions. Then

1. f^k for $k \geq 1$ is measurable.
2. $f + g$, and $f \cdot g$ are measurable.

Proof. (1) for k odd, $\{f^k > a\} = \{f > a^{1/k}\}$. For k even, $\{f^k > a\} = \{f > a^{1/k}\} \cup \{f < -a^{1/k}\}$. These right hand sets are all measurable.

(2) $\{f + g > a\} = \bigcup_{q \in \mathbb{Q}} \{f > a - r\} \cap \{g > r\}$ $f \cdot g = \frac{1}{4}((f+g)^2 + (f-g)^2)$. $\square$

If f is measurable, and $f(x) = g(x)$ for all $x \notin E$, where $m(E) = 0$, then $g(x)$ is measurable as well. We say that $f = g$ a.e..

Approximation by simple and step functions

Theorem 5. *Let f be a nonnegative measurable function on $\mathbb{R}^d$. Then there exists a sequence $\varphi_n \nearrow f$ pointwise such that φ_n are simple functions.*

Proof. Truncate first. Choose Q_k cubes of side length k, and define $F_k(x) = f(x) \wedge k$ if $x \in Q_k$ and 0 otherwise. Clearly $F_k(x) \to f(x)$ as $k \to \infty$, and the range of $F_k(x)$ is $[0,k]$. For any k, j define

$$E_{l,j} = \left\{ x \in Q_k \mid \frac{l}{j} \le F_k(x) \le \frac{l+1}{j} \right\} \quad 0 \le l \le kj$$

Moreover, define $F_{k,j} = \sum_{l=0}^{kj} \frac{l}{j} \chi_{E_{l,j}}(x)$ and let $\varphi_k = F_{k,k}$. Choose the subsequence $\phi_m = \varphi_{2^m}$, and note that $\phi_m \nearrow f$ so we are done. □

Theorem 6. *Let f be measurable on $\mathbb{R}^d$. Then there exists a sequence $(\varphi_n)_n$ of simple functions such that $|\varphi_n(x)|$ is increasing, and $\lim_k \varphi_k(x) = f(x)$ pointwise.*

Proof. Note $f(x) = f^+(x) - f^-(x)$, where $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$. These two functions are measurable and nonnegative, so we can choose $\varphi_n^{(1)} \nearrow f^+$ and $\varphi_n^{(2)} \nearrow f^-$. Then $\varphi_n(x) := \varphi_n^{(1)} - \varphi_n^{(2)}$ is our desired sequence that converges pointwise to f, and satisfies $|\varphi_n(x)|$ is increasing. □

Definition 0.4. A **step function** is a simple function where the E_k are rectangles.

Theorem 7. *Let f measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions that converges to f almost everywhere.*

Lecture 6: Properties of Measurable Functions (9/19)

Proof (of previous). Without loss of generality, $f = \chi_E$, where $E \subseteq \mathbb{R}^d$ is measurable with finite measure. For all $\epsilon > 0$, there exists closed cubes $Q_1, \ldots, Q_n$ such that $m(E \Delta \bigcup_{j=1}^n Q_j) < \epsilon$. Moreover, we can decompose $\bigcup_{j=1}^n Q_j = \bigcup_{j=1}^m \tilde{R}_j$, for $\tilde{R}_j$ almost disjoint rectangles. Thus, we can choose R_j disjoint rectangles with $m(E \Delta (\bigcup_{j=1}^N R_j)) < 2\epsilon$.

Thus, $\chi_E(x) = f(x) = \sum_{j=1}^N \chi_{R_j}(x)$ except on a set of measure $\le 2\epsilon$. So, for all $k \ge 1$, there exist $\psi_k(x)$ step functions such that for $E_k = \{x \mid f(x) \ne \psi_k(x)\}$, $m(E_k) < (1/2)^k$. We claim that $\psi_k(x) \to f(x)$ a.e Define $F_k = \bigcup_{j=k+1}^\infty E_j$ and $F = \bigcap_{k=1}^\infty F_k$. In fact, $\psi_k(x) \to f(x)$ for all $x \notin F$. This is because for $x \notin F$, then $x \in \bigcup_k F_k^C$, so there exists a k_0 such that $x \in F_{k_0}^C = \bigcap_k E_j^C$. Thus, for this x, $\psi_k(x) = f(x)$ for all $k \ge k_0$. □

Now, we consider results related to Littlewood's principles. The following theorem essentially says that every convergent sequence of functions is nearly uniformly convergent.

Theorem 8 (Egorov)**.** *Let f_k be a sequence of measurable functions on a set E such that $m(E) < \infty$. Assume $f_k \to f$ a.e. on E. Then for all $\epsilon > 0$, there exists a closed subset $A_\epsilon \subseteq E$ such that $m(E \setminus A_\epsilon) < \epsilon$.*

Proof. Fill in later. $\square$

Theorem 9 (Lusin)**.** *Assume f is a measurable function value on D with $m(E) < \infty$. Then for all $\epsilon > 0$, the closed set $F_\epsilon \subseteq E$, $m(E \setminus F_\epsilon) < \epsilon$ and $f\,|_{F_\epsilon}$ is continuous.*

Proof. Fill in later. $\square$

Lecture 7: Brunn-Minkowski, Integration (9/24)

Muscalu's advice: learn PDEs, since it brings a lot of different fields of math together, and gives motivation for studying mathematics. Consider why you wish to study mathematics.

Theorem 10 (Brunn-Minkowski inequality)**.** *Let $A, B \subseteq \mathbb{R}^d$ measurable sets such that $A + B$ is also measurable. Then*

$$m(A+B)^{1/d} \geq m(A)^{1/d} + m(B)^{1/d} \tag{1}$$

Proof. We prove the claim in cases.

Case 1: Say both A and B are rectangles, with sidelengths a_j and b_j, $j = 1, \ldots, d$. Then (1) becomes

$$\Big(\prod_{j=1}^{d} a_j\Big)^{1/d} + \Big(\prod_{j=1}^{d} b_j\Big)^{1/d} \leq \Big(\prod_{j=1}^{d} (a_j + b_j)\Big)^{1/d}$$

$$\iff \Big[\prod_{j=1}^{d} (\frac{a_j}{a_j + b_j})\Big]^{1/d} + \Big[\prod_{j=1}^{d} (\frac{b_j}{a_j + b_j})\Big]^{1/d} \leq 1$$

We see that this inequality holds since

$$\begin{aligned}\Big[\prod_{j=1}^{d} (\frac{a_j}{a_j + b_j})\Big]^{1/d} + \Big[\prod_{j=1}^{d} (\frac{b_j}{a_j + b_j})\Big]^{1/d} &\leq \frac{1}{d}\Big[\sum_{j=1}^{d} \frac{a_j}{a_j + b_j} + \sum_{j=1}^{d} \frac{b_j}{a_j + b_j}\Big] \\ &= 1\end{aligned}$$

Where we use the inequality $(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n}$.

Case 2: Say A and B are unions of finitely many rectangles whose interiors are disjoint. We use induction on the total number of rectangles making up A and B. Note that the above was our base case.

We can translate $A \leftarrow A + t_1$ and $B \leftarrow B + t_2$ as needed due to translation invariance of Lebesgue measure. Pick a pair of rectangles (R_1, R_2) in A. There exists $j \in [d]$ such that $R_1 \subseteq A \cap \{x_j < 0\} =: A_-$ and $R_2 \subseteq A \cap \{x_j \geq 0\} =: A_+$ because the rectangles have disjoint interiors.

We can translate B such that if $B_- := B \cap \{x_j < 0\}$ and $B_+ := B \cap \{x_j \geq 0\}$, we have

$$\frac{m(B_\pm)}{m(B)} = \frac{m(A_\pm)}{m(A)}$$

Note that

$$
\begin{aligned}
A+B &\supseteq (A_+ + B_+) \cup (A_- + B_-) \\
m(A+B) &\geq m(A_+ + B_+) + m(A_- + B_-) \\
&\geq \Big(m(A_+)^{1/d} + m(B_+)^{1/d}\Big)^{1/d} + \Big(m(A_-)^{1/d} + m(B_-)^{1/d}\Big)^{d} \quad \text{by induction} \\
&= m(A_+)\Big[1 + \Big(\frac{m(B)}{m(A)}\Big)^{1/d}\Big]^d + m(A_-)\Big[1 + \Big(\frac{m(B)}{m(A)}\Big)^{1/d}\Big]^d \\
&= \Big[1 + \Big(\frac{m(B)}{m(A)}\Big)^{1/d}\Big]^d m(A) \\
&= \Big[m(A)^{1/d} + m(B)^{1/d}\Big]^d
\end{aligned}
$$

Case 3: Suppose A and B are both open sets.

For all $\epsilon > 0$, there exist $A_\epsilon \subseteq A$ and $B_\epsilon \subseteq B$, both unions of almost disjoint rectangles, such that $m(A) \leq m(A_\epsilon) + \epsilon$ and $m(B) \leq m(B_\epsilon) + \epsilon$. Then $A + B \supseteq A_\epsilon + B_\epsilon$ so

$$m(A+B)^{1/d} \geq (m(A_\epsilon) + m(B_\epsilon))^{1/d} \geq m(A_\epsilon)^{1/d} + m(B_\epsilon)^{1/d}$$

by the above case. Taking $\epsilon \to 0$ finishes this case.

Case 4: Suppose A and B are compact. Define $A_\epsilon = \{x \mid d(x, A) < \epsilon\}$ and analogously for B_ϵ. Then $A_\epsilon \searrow A$ and $B_\epsilon \searrow B$. Also,

$$A + B \subseteq A_\epsilon + B_\epsilon \subseteq (A+B)_{2\epsilon}$$

Use case 3 and take $\epsilon \to 0$.

Case 5: Let A and B be generable measurable sets with $A+B$ measurable.

Then there are $A(\epsilon)$ and $B(\epsilon)$ compact sets inside A and B respectively such that $m(A) \leq m(A(\epsilon)) + \epsilon$ and $m(B) \leq m(B(\epsilon)) + \epsilon$. Thus, we have that

$$m(A+B)^{1/d} \geq m(A(\epsilon) + B(\epsilon))^{1/d} \geq m(A(\epsilon))^{1/d} + m(B(\epsilon))^{1/d} \geq (m(A)+\epsilon)^{1/d} + (m(B)+\epsilon)^{1/d}$$

so we are done by taking $\epsilon \to 0$. □

Integration theory

We define the integral in 4 steps.

1. Simple functions
2. Bounded functions supported on sets of finite measure
3. Nonnegative functions
4. General case

1. Simple functions

First, consider a simple function $f(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$, where the $a_k \in \mathbb{R}^d$, and the E_k are measurable and finite measure. We can assume E_k disjoint without loss of generality. We define the **Lebesgue integral** of f as

$$\int_{\mathbb{R}^d} f(x)\, dx := \sum_{k=1}^{N} a_k m(E_k)$$

Proposition 10.

1. The integral is independent of the representation of the simple function.
2. Linearity:
$$\int \alpha f + \beta g \, dx = \alpha \int f \, dx + \beta \int g \, dx$$
3. Monotonicity:
$$\varphi \leq \psi \implies \int \varphi \, dx \leq \int \psi \, dx$$
4. Additivity: If E and F are disjoint,
$$\int_{E \cup F} f \, dx = \int_E f \, dx + \int_F f \, dx$$
where we define $\int_E f \, dx = \int_{\mathbb{R}^d} f(x) \chi_E(x) \, dx$
5. Triangle inequality:
$$\left| \int f \, dx \right| \leq \int |f| \, dx$$

2. Bounded functions supported on a set of finite measure

Note that for f measurable, $\operatorname{supp}(f) = \{x \mid f(x) \neq 0\}$ is measurable. We are considering bounded measurable functions f with $m(\operatorname{supp}(f)) < \infty$.

Recall that there exist simple functions $(\varphi_n)_n$ such that $\varphi_n(x) \to f(x)$ for all x.

Lemma 3. *Let f bounded and supported on E with $m(E) < \infty$. Let also $(\varphi_n)_n$ as before and also bounded. Then*

1. $\lim_{n \to \infty} \int \varphi_n(x) \, dx$ *exists.*
2. *If $f = 0$ a.e, then* $\lim_{n \to \infty} \int \varphi_n(x) \, dx = 0$

Proof. Denote $I_n := \int \varphi_n(x) \, dx$. It suffices to prove that $(I_n)_n$ is Cauchy. Let $\epsilon > 0$. Egorov implies that for all $\tilde{\epsilon} > 0$, there exists $A_{\tilde{\epsilon}} \subseteq E$ such that $\varphi_n \to f$ uniformly in $A_{\tilde{\epsilon}}$ and $m(E \setminus A_{\tilde{\epsilon}}) < \tilde{\epsilon}$. Thus, we have

$$\begin{aligned} |I_n - I_m| &\leq \int_E |\varphi_n(x) - \varphi_m(x)| \, dx \\ &\leq \int_{A_{\tilde{\epsilon}}} |\varphi_n - \varphi_m| \, dx + \int_{E \setminus A_{\tilde{\epsilon}}} |\varphi_n - \varphi_m| \, dx \end{aligned}$$

Then we use uniform convergence. □

Lecture 8: Lebesgue Integral (9/26)

In light of the previous lemma, we define for f bounded and supported on a set of finite measure the integral

$$\int_{\mathbb{R}^d} f(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n(x)\,dx$$

where $\varphi_n(x) \to f(x)$ are simple functions. The choice of simple functions does not matter.

Theorem 11 (Bounded convergence)**.** *Let $(f_n)_n$ be measurable, bounded by M, and supported on E of finite measure.*

If $f_n(x) \to f(x)$ a.e*, then f is measurable, supported on E, and*

$$\lim_{n\to\infty} \int \left|f_n(x) - f(x)\right|\,dx \to 0$$

The following theorem states that any Riemann integrable function is Lebesgue integrable.

Theorem 12. *Let f be Riemann integrable on $[a,b]$ (so f is bounded in particular). Then*

$$R\int_{[a,b]} f(x)\,dx = L\int_{[a,b]} f(x)\,dx$$

where $R\int$ is the Riemann integral and $L\int$ is the Lebesgue integral.

Proof. There exist $(\varphi_k)_k$ and $(\psi_k)_k$ simple step functions such that φ_k are increasing and ψ_k are decreasing, and $\varphi_k \le f \le \psi_k$. Then we have that

$$\lim_k R\int \varphi_k\,dx = \lim_k R\int \psi_k\,dx = R\int f\,dx$$

note that these are the lower and upper Riemann sums. By definition of the Lebesgue integral on simple functions,

$$R\int \varphi_k\,dx = L\int \varphi_k\,dx$$

and similarly for ψ_k. Define $\tilde{\varphi}$ such that $\varphi \nearrow \tilde{\varphi}_k$ and $\tilde{\psi}$ such that $\psi_k \searrow \tilde{\psi}$. Both are measurable by bounded convergence. Then

$$\lim_k L\int \varphi_k\,dx = L\int \tilde{\varphi}\,dx$$

And similarly for ψ_k. Then we have that

$$L\int [\tilde{\psi}(x) - \tilde{\varphi}(x)\,dx = 0$$

This together with the fact that $\tilde{\psi}(x) - \tilde{\varphi}(x) \ge 0$ gives us that $\tilde{\psi}(x) = \tilde{\varphi}(x)$ a.e (see below). Thus, we have that $\tilde{\psi}(x) = \tilde{\varphi}(x) = f(x)$ a.e. Since $\varphi_k \to f$ a.e and $\psi_k \to f$ a.e, we have that

$$R\int f\,dx = \lim_k R\int \varphi_k\,dx = \lim_k L\int \varphi_k\,dx = L\int f\,dx$$

$\square$

To see that $\int g\,dx = 0$ for a nonnegative function $g \geq 0$ implies that $g = 0$ a.e, consider

$$E_k = \{x \mid g(x) > 1/k\}$$

which are increasing and measurable. We use Chebyshev's inequality, which states that

$$m(\{x \mid |f(x)| > a\}) \leq \frac{1}{a} \int |f(x)|\, dx$$

To see this, let $A = m(\{x \mid |f(x)| > a\})$, and note that

$$m(A) = \int \chi_A\, dx = \frac{1}{a} \int a\chi_A\, dx \leq \frac{1}{a} \int |f(x)|\chi_A\, dx \leq \frac{1}{a} \int |f(x)|\, dx$$

Applying Chebyshev gives us that

$$m(E_k) \leq k \int_a^b g\, dx = 0$$

Then $m(E_k) \nearrow m(\{g > 0\}) = 0$.

Lebesgue integral for nonnegative functions

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function that can also take on extended values. Define

$$\int_{\mathbb{R}^d} f\, dx = \sup_{\substack{g \\ 0 \leq g \leq f \\ g \in S}} \int_{\mathbb{R}^d} g\, dx$$

where S is the set of all bounded measurable functions supported on a set of finite measure. If this integral is finite, we say f is Lebesgue measurable.

Lebesgue integral for general functions

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function that can also take on extended values. We say f is Lebesgue integrable if $|f|$ is Lebesgue integrable. Let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. If f is Lebesgue integrable, we define the integral

$$\int_{\mathbb{R}^d} f\, dx = \int_{\mathbb{R}^d} f^+\, dx - \int_{\mathbb{R}^d} f^-\, dx$$

Lemma 4 (Fatou)**.** *Let $(f_n)_n \geq 0$ and measurable. If $f_n \to f$ a.e, then*

$$\int f\, dx \leq \liminf_n \int f_n\, dx$$

Note that replacing the $\liminf$ by a limit does not give a true statement.

Example 0.6. Define the functions

$$f_n(x) = \begin{cases} n & x \in [0, 1/n] \\ 0 & \text{otherwise} \end{cases}$$

Then we have that $f_n(x) \to 0$ a.e, but $\int f_n\, dx \to 1$.

Proof of Fatou's lemma. Let $0 \le g \le f$ where g bounded and supported on a set of finite measure. Then define $g_n = g \wedge f_n$, and note that $g_n \to g$ a.e. Then bounded convergence says

$$\liminf_n \int g_n \, dx = \lim_n \int g_n \, dx = \int g \, dx \le \liminf_n \int f_n \, dx$$

$\square$

Corollary 3. If $(f_n)_n \ge 0$ measurable, $f_n \le f$, and $f_n \to f$ a.e,

$$\lim_n \int f_n \, dx = \int f \, dx$$

Corollary 4 (Monotone convergence)**.** Let $(f_n)_n$ measurable and $f_n \nearrow f$. Then

$$\int f_n \, dx \nearrow \int f \, dx$$

Proposition 11. Let f be integrable on $\mathbb{R}^d$. Then for all $\epsilon > 0$,

1. There exists a B of finite measure such that

$$\int_{B^c} |f| \, dx < \epsilon$$

2. (Absolute continuity) There exists $\delta > 0$ such that for all E with $m(E) < \delta$,

$$\int_E |f(x)| \, dx < \epsilon$$

Lecture 9: Lebesgue Integration Properties (10/1)

Suppose φ is a real function that is nice near zero but has tails that decay at the rate $1/x$. Then this is not Riemann or Lebesgue integrable. However, we can still integrate $\varphi(x)e^{ix}$ by integration by parts. We do not just study Lebesgue integrable functions.

In our example sequence of functions f_n from last lecture, such that $f_n(x) \to 0$ but $\int f_n \, dx \to 1$, another way to view the limit of the f_n is as a distribution—the Dirac delta distribution.

Proof of Proposition 11. 1. Assume $f \ge 0$ for ease of notation. Let $B_n = B(0,n) = \{x \mid |x| \le n\}$. Let $f_n = f \mathbb{1}_{B_n}$. Then $f_n \nearrow f$ pointwise. Thus, by monotone convergence,

$$\lim_n \int_{\mathbb{R}^d} f_n \, dx = \int_{\mathbb{R}^d} f \, dx$$

$$0 < \int f \, dx - \int f_n \, dx < \epsilon$$

for large enough n. Since we have that $\int f_n \, dx = \int f \mathbb{1}_{B_n} \, dx$, we know that

$$\int f \, dx - \int f_n \, dx = \int f \mathbb{1}_{B_n^c} \, dx = \int_{B_n^c} f \, dx$$

so we are done.

2. Define $f_n = f\mathbb{1}_{E_n}$, where $E_n = \{f \le n\}$. Clearly $f_n \nearrow f$ and there exists N large such that

$$\int f - f_n \, dx < \frac{\epsilon}{2}$$

Thus, we have

$$\begin{aligned}\int_E f \, dx &= \int_E f - f_N \, dx + \int_E f_N \, dx \\ &\le \frac{\epsilon}{2} + N \cdot m(E)\end{aligned}$$

so, for any measurable E with $m(E) < \frac{\epsilon}{2N}$, $\int_E f \, dx < \epsilon$. ☐

Theorem 13 (Dominated convergence)**.** *Let f_n measurable such that $f_n \to f$ a.e. If $f_n \le g$ a.e for some g integrable, then*

$$\int |f_n - f| \, dx \to 0$$

Proof. Let $\epsilon > 0$. Let $E_n = \{x : |x| \le n, \quad g(x) \le n\}$. Note that E_n has finite measure, so by the reasoning used in the last proof, $\int_{E_n^c} g \, dx < \epsilon$ for N large. Then, $f_n \mathbb{1}_{E_n}$ is bounded and supported on a set of finite measure, so by bounded convergence,

$$\int_{E_n} |f_n - f| \, dx < \epsilon$$

for large $n \ge N$ for some $N \in \mathbb{N}$. We can split the total integral so that

$$\begin{aligned}\int_{\mathbb{R}^d} |f_n - f| \, dx &= \int_{E_n} |f_n - f| \, dx + \int_{E_n^c} |f_n - f| \\ &\le \int_{E_n} |f_n - f| \, dx + 2\int_{E_n^c} |g| \, dx \\ &< 3\epsilon\end{aligned}$$

for $n \ge N$. ☐

$L^1(\mathbb{R}^d)$ is the space of Lebesgue integrable functions on $\mathbb{R}^d$. It is a vector space, that has the norm

$$\|f\|_1 = \int_{\mathbb{R}^d} |f(x)| \, dx$$

Theorem 14 (Riesz-Fischer)**.** *$L^1(\mathbb{R}^d)$ is complete, and hence is a Banach space.*

Proof. Let $(f_n) \subseteq L^1(\mathbb{R}^d)$ be Cauchy. We can replace f with a subsequence such that $\|f_{n+1} - f_n\|_1 < \frac{1}{2^n}$. Define

$$f(x) = f_1(x) + \sum_{n=2}^{\infty} f_n(x) - f_{n-1}(x)$$

Consider $g(x) = |f_1| + \sum_{n=2}^{\infty} |f_n - f_{n-1}|$. We will show that g is integrable so that dominated convergence applies. g is integrable since

$$\begin{aligned}\|g\| &\le \|f_1\| + \left\| \sum_{n=2}^{\infty} |f_n - f_{n-1}| \right\| \\ &= \|f_1\| + \sum_{n=2}^{\infty} \|f_n - f_{n-1}\| \quad \text{monotone convergence} \\ &< \|f_1\| + 2\end{aligned}$$

Since $f_n \to f$, dominated convergence gives that $f_n \to f$ in L^1. $\square$

Corollary 5. if $\|f_n - f\|_1 \to 0$, then there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ a.e.

Proof. To see this, take a subsequence as in the proof of Riesz-Fischer. $\square$

Corollary 6. All of the following families of functions are dense in $L^1(\mathbb{R}^d)$.

- Simple functions
- Step functions
- Continuous function with compact support

Proof. Let $f \in L^1(\mathbb{R}^d)$. Suppose $f \geq 0$.

1. There there are increasing simple functions φ_n such that $\varphi \to f$ pointwise. By dominated convergence, $\varphi \xrightarrow{L^1} f$.

2. Let $E \subseteq \mathbb{R}^d$ measurable. We want step functions such that

$$\|\mathbb{1}_E - \psi_n\|_1 < \epsilon$$

Choose R_j almost disjoint rectangles such that

$$m\Big(E \Delta \bigcup_{j=1}^{n} R_j\Big)$$

then let $\psi_n = \sum_{j=1}^{n} \mathbb{1}_{R_j}$

3. We can take a sequence of continuous functions that are step functions with continuous bridges that get decrease. $\square$

Proposition 12. Let $f \in L^1(\mathbb{R}^d)$. Then $\|f(x+h) - f(x)\|_1 \to 0$ as $h \to 0$.

Proof. Let $f_h = f(x+h)$, and choose $\epsilon > 0$. Pick g continuous such that $\|f - g\|_1 < \epsilon$. Then

$$\begin{aligned}\|f_h - f\| &\leq \|g_h - g\| + \|f_h - g\| + \|f - g\| \\ &< 3\epsilon\end{aligned}$$

$\square$

Lecture 10: Fubini/ Tonelli (10/3)

Fubini's theorem

Let $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where $d_1 + d_2 = d$. Consider a function $f : \mathbb{R}^d \to \mathbb{R}$. Define $f^y : \mathbb{R}^{d_1} \to \mathbb{R}$ by $f^y(x) = f(x,y)$ and for a subset E of $\mathbb{R}^d$, let $E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$.

Theorem 15. *Let $f(x,y)$ be integrable over $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then, for almost every $y \in \mathbb{R}^{d_2}$,*

1. *The slice f^y is integrable on $\mathbb{R}^{d_1}$*
2. *The function $y \mapsto \int f^y(x)\,dx$ is integrable over $\mathbb{R}^{d_2}$*
3. $$\int_{\mathbb{R}^{d_2}} \Big(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \Big)\,dy = \int_{\mathbb{R}^d} f(z)\,dz$$

The analogous statements hold for x.

Proof. Denote by $\mathcal{F}$ the set of all integrable functions in $\mathbb{R}^d$ satisfying 1.,2.,3.. We want to show that $L^1(\mathbb{R}^d) \subseteq \mathcal{F}$.

(1) First, we note that any finite linear combination of functions in $\mathcal{F}$ is in $\mathcal{F}$.

(2) Now, we want to show that if $(f_k)_k$ is a sequence of measurable functions in $\mathcal{F}$ such that $f_k \nearrow f$ or $f_k \searrow f$, where f is integrable, then $f \in \mathcal{F}$. Without loss of generality, the f_k are nonnegative. Thus, dominated convergence says

$$\int_{\mathbb{R}^d} f_k(x,y)\,dx\,dy \to \int_{\mathbb{R}^d} f(x,y)\,dx\,dy$$

For all k, there exists an $A_k \subseteq \mathbb{R}^{d_2}$ with $m(A_k) = 0$ such that f_k^y are integrable for $y \notin A_k$. Consider $A = \bigcup_{k=1}^{\infty} A_k$, which also has measure zero. By monotone convergence, if $y \in \mathbb{R}^{d_2} \setminus A$, then we have the convergence

$$g_k(y) = \int f_k^y\,dx \nearrow g(y) = \int_{\mathbb{R}^{d_1}} f^y(x)\,dx$$

and also

$$\int_{\mathbb{R}^{d_2}} g_k(y)\,dx \to \int_{\mathbb{R}^{d_2}} g(y)\,dy$$

but $f_k \in \mathcal{F}$ and $\int_{\mathbb{R}^{d_2}} g_k(y)\,dy = \int_{\mathbb{R}^d} f_k(x,y)\,dx\,dy$ so that

$$\int_{\mathbb{R}^{d_2}} g(y)\,dy = \int_{\mathbb{R}^d} f(x,y)\,dx\,dy$$

and since f is integrable in $\mathbb{R}^d$, we have that g is integrable over $\mathbb{R}^{d_2}$.

(3) Now, we show that if $E \in \mathbb{R}^d$ with $m(E) < \infty$ that is a G_δ, then $\mathbb{1}_E \in \mathcal{F}$. We prove this in subcases.

(3a) Suppose $E = Q_1 \times Q_2$ open cubes. Then we have that for all $y \in \mathbb{R}^{d_2}$, $\mathbb{1}_E(x,y)$ is measurable in x. This is because $\mathbb{1}_E(x,y) = \mathbb{1}_{Q_1}\mathbb{1}_{Q_2}$. Note that

$$g(y) = \int_{\mathbb{R}^{d_1}} \mathbb{1}_E(x,y)\,dx = \begin{cases} |Q_1| & y \in Q_2 \\ 0 & \end{cases}$$

so $g = |Q_1|\,\mathbb{1}_{Q_2}$ and thus

$$\int_{\mathbb{R}^{d_2}} g(y)\,dy = |Q_1||Q_2| = \int_{\mathbb{R}^d} \mathbb{1}_E(x,y)\,dx\,dy$$

(3b) Now, say that $E \subseteq \partial Q$, i.e. that E is contained in the boundary of a cube. Note that $m(E) = 0$ and that $\int_{\mathbb{R}^d} \mathbb{1}_E(x,y)\,dx\,dy = m(E) = 0$. Moreover, for almost every y, we have that E^y has measure zero in $\mathbb{R}^{d_1}$, where ....

(3c) Suppose $E = \bigcup_{k=1} Q_k$ are closed rentangles with dijoint supports.

(3d) Let $E = \mathcal{O}$ be an open set of finite measure. Choose Q_j a collection of almost disjoint cubes.

(4d) Let $E = G_\delta$. We want $\mathbb{1}_E \in \mathcal{F}$. Then we define $E = \bigcap_{k=1}^{\infty} \prime_k$ for open sets.

$\square$

Theorem 16 (Tonelli)**.** *Suppose $f(x,y)$ is a non-negative measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:*

1. *$x \mapsto f(x,y)$ is measurable on $\mathbb{R}^{d_1}$*
2. *$y \mapsto \int_{\mathbb{R}^{d_1}} f(x,y)\, dx$ is measurable on $\mathbb{R}^{d_2}$*
3. $$\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} f(x,y)\, dx = \int_{\mathbb{R}^d} f(x,y)\, dx\, dy$$

Corollary 7. Let $E \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be measurable. Then for a.e $y \in \mathbb{R}^{d_2}$, $E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$ is measurable. Moreover, $y \mapsto m(E^y)$ is measurable and

$$m(E) = \int_{\mathbb{R}^{d_2}} m(E^y)\, dy$$

Note that $E = [0,1] \times \mathcal{N}$ provides an example where E^y is measurable for each y, but E is not measurable, where $\mathcal{N}$ is a non-measurable subset of $\mathbb{R}$.

Proposition 13. If $E_1 \subseteq \mathbb{R}^{d_1}$ and $E_2 \subseteq \mathbb{R}^{d_2}$ are measurable, then $E = E_1 \times E_2$ is measurable in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and

$$m(E) = m(E_1)m(E_2)$$

Lemma 5.

$$m_*(E_1 \times E_2) \le m_*(E_1)m_*(E_2)$$

Lecture 11: Integration, Differentiation (10/8)

Suppose we have $\left(\sum_{l\in\mathbb{Z}} |a_l|^2\right)^{1/2} < \infty$. We want to say something like

$$\int_a^b \sum_{n\in\mathbb{Z}} a_n e^{inx}\, dx =^? \sum_{n\in\mathbb{Z}} a_n \int_a^b e^{inx}\, dx$$

The term on the right is absolutely convergent, but in general Riemann integration cannot handle the left series integrand. Note that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} a_n \int_a^b e^{inx}\, dx &\le \sum_{n\neq 0} C|a_n| \frac{1}{|n|} + (b-a)|a_0| \\
&\le C\Big(\sum_{n\neq 0} |a_n|^2\Big)^{1/2} + (b-a)|a_0| && \text{Cauchy-Schwarz} \\
&< \infty
\end{aligned}$$

Recall that $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are Banach spaces. In fact, $L^2(\mathbb{R})$ is a Hilbert space. Note that $F_N = \sum_{n=-N}^{N} a_n e^{inx}$ is in L^2. Let $N > M$. Then we have (working over $L^2[0,2\pi]$ and using that $L^2[a,b]$ is Banach),

$$\|F_N - F_M\|_2^2 = \left\| \sum_{M<|n|\le N} a_n e^{inx} \right\|_2^2 = \sum_{M<|n|\le N} |a_n|^2 \to 0$$

So that F_N is Cauchy and hence has a limit F in $L^2[0,2\pi]$. We want to know whether

$$\int_a^b F_n(x)\, dx \to \int_a^b F(x)\, dx$$

This does in fact holds, since (assuming $a,b \in (0,2\pi)$)

$$\begin{aligned}
\left| \int F_n \, dx - \int F \, dx \right| &\le \int_a^b |F - F_N| \, dx \\
&\le \Big(\int_0^{2\pi} |F(x) - F_N(x)|^2 \, dx \Big)^{1/2} (b-a)^{1/2} && \text{Cauchy-Schwarz} \\
&= \|F_N - F\|_2 (b-a)^{1/2} \to 0
\end{aligned}$$

Recall that $l^2 = \{(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} : \Big(\sum_{n\in\mathbb{N}} |a_n|^2\Big)^{1/2} < \infty\}$ is a Hilbert space. If we have that $(a_n)_n \in l^2$, then $\sum_{n\in\mathbb{Z}} a_n e^{inx} \in L^2[0,2\pi]$. Conversely, if $f \in L^2[0,2\pi]$, then the sequence of Fourier coefficients satsifies $(\hat{f}(n))_n \in l^2$, where $\hat{f}(x) = \int_0^{2\pi} f(x) e^{-inx}\, dx$.

Say we have $f \in L^2[0,2\pi]$. Then defining $S_N f(x) = \sum_{|n|\le N} \hat{f}(n) e^{inx}$, we can show that $S_N \to f$ in L^2. Let $\epsilon > 0$. The smooth functions are dense in L^2, so that there is a smooth g with $\|g - f\|_2 < \epsilon$. We have that $S_N g \to g$ uniformly by undergraduate Fourier analysis. Thus, we have

$$\|S_N f - f\|_2 \le \|S_N f - S_N g\| + \|S_N g - g\| + \|g - f\|$$

The rightmost two summands are already controlled. Now, note that

$$\begin{aligned}
\|S_N f - S_N g\|_2 &= \|S_N(f-g)\|_2 \\
&\le \|f - g\|_2 < \epsilon
\end{aligned}$$

This is because

$$\begin{aligned}
S_N(h) &= \sum_{|n|\le N} \langle h, e^{inx} \rangle e^{inx} \\
\|S_N(h)\|_2^2 &= \sum_{|n|\le N} \left| \langle h, e^{inx} \rangle \right|^2 \\
&\le \|h\|_2^2 = \langle h, h \rangle
\end{aligned}$$

Thus, we have proven $S_N f \to f$ in L^2. As a corollary,

$$\|f\|_2 = \Big(\sum_{n\in\mathbb{Z}} \left| \hat{f}(n) \right|^2 \Big)^{1/2}$$

Differentiation and Integration

Say f is Lebesgue integrable on $[a,b]$, and consider

$$F(x)=\int_a^x f(y)\,dy$$

Is it true that F is differentiable almost everywhere and $F'=f$ on some set? This turns out to be true.

Another thing to consider is which conditions on F on $[a,b]$ guarantee that F' exists a.e, that F' is integrable, and that the fundamental theorem of calculus holds

$$F(b)-F(a)=\int_a^b F'\,dx$$

For $F=\int_a^x f(y)\,dy$,

$$\begin{aligned}\frac{F(x+h)-F(x)}{h}&=\frac{1}{h}\int_x^{x+h} f(y)\,dy\\&=\frac{1}{|I|}\int_I f(y)\,dy \qquad\qquad \text{where } I \text{ an interval with } x\in I\end{aligned}$$

we want to know whether this quantity coverges to $f(x)$ as $h\to 0$ or $|I|\to 0$. More generally, for cubes B, we consider

$$\lim_{\substack{|B|\to\infty\\ x\in B}}\frac{1}{|B|}\int_B f(y)\,dy$$

Let $f\in L^1(\mathbb{R})$. We want

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(y)\,dy=f(x)\text{ a.e}\\ \iff&\lim_{h\to 0}\Big[\frac{1}{h}\int_x^{x+h} f(y)\,dy-f(x)\Big]=0\text{ a.e}\\ \iff&\limsup_{h\to 0}\left|\frac{1}{h}\int_x^{x+h} f(y)-f(x)\,dy\right|=0\text{ a.e}\end{aligned}$$

Call this final lim sup quantity $\mathcal{E}(f)(x)$. We want to prove $m(\{\mathcal{E}(f)>\lambda\})=0$ for all $\lambda>0$. We define for F measurable,

$$\|F\|_{1,\infty}=\sup_{\lambda>0}\lambda\, m(|F|>\lambda)$$

This is the **weak-L^1** norm, a quasinorm (triangle inequality does not hold). Recall that Chebyshev's inequality states that $\|F\|_{1,\infty}\le\|F\|_1$. We have that $\|F+G\|_{1,\infty}\le C(\|F\|_{1,\infty}+\|G\|_{1,\infty})$ for some $C>1$. Note that what we wish to prove above is that $\|\mathcal{E}(f)\|_{1,\infty}=0$. Let $\epsilon>0$, and let g smooth such that $\|f-g\|_1<\epsilon$. Then by the triangle inequality we have

$$\|\mathcal{E}(f)\|_{1,\infty}\le C\Big[\|\mathcal{E}(g)\|_{1,\infty}+\epsilon+\|M(f-g)\|_{1,\infty}\Big]$$

Where we define, where Q are cubes,

$$Mh(x)=\sup_{Q:x\in Q}\frac{1}{|Q|}\int_I |h(y)|\,dy$$

We have $\|\mathcal{E}(g)\|_{1,\infty}=0$ since g is smooth, so we need only control the last term. M, the **Hardy-Littlewood maximal function**, arises naturally. We will show that $M:L^1\to L^{1,\infty}$ is bounded.

Lecture 12: Maximal Function, Density (10/10)

Let $L^{1,\infty}$ denote the set of functions with finite weak L^1 norm. Note that $\frac{1}{|x|} \in L^{1,\infty}$ but $\frac{1}{|x|} \notin L^1$. $L^{1,\infty}$ is indeed strictly smaller than L^1.

Theorem 17 (Properties of Hardy-Littlewood maximal function)**.** *Let $f \in L^1(\mathbb{R}^d)$. Then*

1. *Mf is measurable*
2. *$Mf(x) < \infty$ a.e*
3. *$\|Mf\|_{1,\infty} \le C\|f\|_1$ i.e. M maps L^1 to $L^{1,\infty}$*

Proof. (1.) Let $\alpha > 0$, and consider $\{x \in \mathbb{R}^d : Mf(x) > \alpha\}$. This is an open set, so Mf is measurable.
(2.) Note that $\{x : Mf(x) = \infty\} \subseteq \{x : Mf(x) > \alpha\}$. We will show that

$$m(\{x : Mf(x) > \alpha\}) \le \frac{b}{\alpha}\|f\|_1 < \infty \qquad \text{some } b \in \mathbb{R}$$

It suffices to show this since taking $\alpha \to 0$ shows that $\{Mf = 0\}$ has measure zero.
(3.) Fix $\alpha > 0$, and consider $E_\alpha = \{Mf > \alpha\}$. Suppose for any $x \in E_\alpha$, there exists a ball $B_x \subseteq \mathbb{R}^d$ such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \alpha$$

Note that $E_\alpha = \bigcup_x B_x$. Choose a compact $K \subseteq E_\alpha$. The goal is to show that $m(K) \le \frac{C}{\alpha}\|f\|_1$. Choose a finite cover $B_1, \dots, B_N$ of K from the B_x. Note that if the B_j were all disjoint, then we are done, because

$$\begin{aligned} m(K) &\le \sum_{j=1}^{N} m(B_j) \\ &\le \sum_j \frac{1}{\alpha} \int_{B_j} |f(y)|\, dy \\ &= \frac{1}{\alpha} \sum_j \|f\|_1 \end{aligned}$$

In fact, by flower petal argument, we can choose disjoint S_j such that $K \subseteq \bigcup_{j=1}^M 3S_j$ from these B_j. This finishes the proof. ∎

As an aside, letting $A_h f(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\, dt$, note that

$$\begin{aligned} f * \frac{1}{2h}\mathbb{1}_{[-h,h]}(x) &= \int_{\mathbb{R}} f(y) \frac{1}{2h} \mathbb{1}_{[-h,h]}(x-y)\, dy \\ &= \frac{1}{2h}\int_{x-h}^{x+h} f(t)\, dt \\ &= A_h(x) \end{aligned}$$

so that ”averaging is convolution with a bump”.

Let the **locally integrable functions** $L^1_{\text{loc}}(\mathbb{R}^d)$ be the set of all measurable f on $\mathbb{R}^d$ such that for every ball B the function $f(x)\mathbb{1}_{B(x)}$ is integrable. Observe that if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then the theorem still holds.

Let $E \subseteq \mathbb{R}^d$ be measurable, and suppose that $f = \mathbb{1}_E$. Then we have that

$$\lim_{\substack{|B|\to 0\\ x\in B}} \frac{m(E\cap B)}{m(B)} = \lim_{\substack{|B|\to 0\\ x\in B}} \frac{1}{m(B)} \int_B \mathbb{1}_E(y)\,dy$$

$x \in \mathbb{R}^d$ is called a point of **Lebesgue density** if the above quantity is equal to 1.

Corollary 8. For any measurable $E \subseteq \mathbb{R}^d$,

1. a.e $x \in E$ is a density point
2. a.e $x \notin E$ is not a density point

The **Lebesgue set** of f is the set of all points $x \in \mathbb{R}^d$ such that $|f(x)| < \infty$ and

$$\lim_{\substack{|B|\to 0\\ x\in B}} \frac{1}{|B|} \int_B |f(y) - f(x)|\,dy = 0$$

Observation 0.6. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then a.e $x \in \mathbb{R}^d$ belongs to the Lebesgue set of f

Proof. Let $r \in \mathbb{Q}$. Then there exists a measurable E_r with $m(E_r) = 0$ such that for all $x \notin E_r$,

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)} \int_B |f(y) - r|\,dy = f(x) - r$$

$E = \bigcup_{r\in\mathbb{Q}} E_r$ has measure zero. Let $\bar{x} \notin E$ such that $|f(\bar{x})| < \infty$. Then for any $\epsilon > 0$, there exists $r \in \mathbb{Q}$ such that $|f(\bar{x}) - r| < \frac{\epsilon}{3}$. Then

$$\frac{1}{m(B)} \int_B |f(y) - f(\bar{x})|\,dy \le \frac{1}{m(B)} \int_B |f(y) - r|\,dy + |f(\bar{x}) - r|$$

Taking $\limsup$ over smaller balls contains $\bar{x}$, we bound the left by ϵ. ☐

Lecture 13: Kernels (10/17)

Recall that for $h > 0$, $\varphi_h(x) = \frac{1}{2h}\mathbb{1}_{[-h,h]}(x)$ has the property that $f * \varphi_h = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt$.

Note that

$$\begin{aligned}\int_{-\infty}^{x} f(t)\,dt &= \int_{\mathbb{R}} f(t)\mathbb{1}_{(-\infty,x)}(t)\,dt \\ &= \int_{\mathbb{R}} f(t)\mathbb{1}_{(0,\infty)}(x-t)\,dt \\ &= (f * \mathbb{1}_{(0,\infty)})(x)\end{aligned}$$

so that the antiderivative is also a convolution.

Good kernels and approximation of the identity

We consider integrating functions with kernels

$$f * K_\delta(x) = \int f(x-y) K_\delta(y)\, dy$$

$(K_\delta)_{\delta>0}$ is called a **good family of kernels** if it satisfies:

- $\int_{\mathbb{R}^d} K_\delta(x)\, dx = 1$ for all $\delta > 0$
- $\int_{\mathbb{R}^d} |K_\delta(x)|\, dx \leq A$ for all $\delta > 0$ for some $A \in \mathbb{R}_{\geq 0}$
- For $a > 0$, $\int_{|x|>a} K_\delta(x)\, dx \to 0$ as $\delta \to 0$

The family of K_δ is called an **approximation of the identity** if in addition to integrating to 1, the K_δ satisfy.

- $|K_\delta(x)| \leq C \frac{1}{\delta^d} \qquad \forall \delta > 0$
- $|K_\delta(x)| \leq C \frac{\delta}{|x|^{d+1}} \qquad \forall \delta > 0$

Note that these two conditions imply the second and third condition, respectively, for a good family of kernels.

Note that discrete convolution is represented as a Toeplitz matrix. Elements of index with constant $i-j$ (along the diagonals) are constant.

Theorem 18. *Let $f \in L^1(\mathbb{R}^d)$ and K_δ an approximation of the identity. Then*

$$f * K_\delta(x) \to f(x) \quad \text{as } \delta \to \infty$$

for every x in the Lebesgue set of f.

Proof. First, note that

$$\begin{aligned} |f * K_\delta(x) - f(x| &= \left| \int_{\mathbb{R}^d} [f(x-y) - f(x)] K_\delta(y)\, dy \right| \\ &\leq \int_{\mathbb{R}^d} |[f(x-y) - f(x)] K_\delta(y)|\, dy \end{aligned}$$

We use the following observation, that follows from the definition of the Lebesgue set

Observation 0.7. Let x be a Lebesgue point of f

$$A(r)f(x) = \frac{1}{r^d} \int_{|y|\leq r} |f(x-y) - f(x)|\, dy$$

Then $A(r)f(x)$ is a continuous function of r, and $A(r) \to 0$. Also, $A(r)f$ is bounded.

Then we have

$$\int_{\mathbb{R}^d} \left|[f(x-y)-f(x)]K_\delta(y)\right| \, dy = \underbrace{\int_{|y|\leq\delta} * \, dy}_{(a)} + \underbrace{\sum_{k=0}^{\infty} 2^k \int_{\delta<|y|\leq 2^{k+1}\delta} * \, dy}_{(b)}$$

$$(a) \leq CA(\delta)$$

so this goes to zero as $\delta \to 0$. Moreover,

$$\begin{aligned}
(b) &\leq C\frac{\delta}{(2^k\delta)^{d+1}} \int_{|y|\leq 2^{k+1}\delta} \left|f(x-y)-f(x)\right| \, dy \\
&\leq \frac{1}{2^k} \frac{C}{(2^{k+1}\delta)^d} \int_{|y|\leq 2^{k+1}\delta} \left|f(x-y)-f(x)\right| \, dy \\
&\leq C\sum_{k=0}^{\infty} \frac{1}{2^k} A(2^{k+1}\delta) \\
&= \sum_{k=0}^{n} * + \sum_{n}^{\infty} * \\
&\lesssim \sum_{k=n}^{\infty} \frac{1}{2^k} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

where $L \lesssim R$ means $L \leq CR$ for some constant C. □

Example 0.7 (Examples of kernels)**.**

- (Poisson kernel) For $y > 0$,

  $$P_y(x) = \frac{1}{\pi} \cdot \frac{y}{x^2+y}$$

  For integrable f,

  $$u(x,y) = f * P_y(x)$$

  satisfies the Laplace equation: $\Delta u(x,y) = 0$.

- (Heat kernel)

  $$H_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}$$

  Consider

  $$u(x,t) = f * H_t(x)$$

  this satisfies the heat equation $u_t = \Delta u$.

- (Poisson kernel)

  $$P_r(\theta) = \begin{cases} \frac{1-r^2}{1-2r\cos(\theta)+r^2} & |\theta| < \pi \\ 0 & |\theta| \geq \pi \end{cases}$$

- (Dirichlet kernel)
$$D_N(x) = \frac{\sin((2N+1)x)}{\sin(x/2)}$$
In this case,
$$f * D_N(x) = S_N f(x)$$
but D_N is not a good kernel (the study of Fourier series is complicated!).

- (Fejer kernel)
$$F_N(x) = \begin{cases} \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)} & |x| \le \pi \\ 0 & |x| > \pi \end{cases}$$

Let $A(h)f(x) = f * \frac{1}{2h}\mathbb{1}_{[-h,h]}(x)$.

- $A(h)f(x) \xrightarrow{L^1} f(x)$ because $A(h)$ maps $L^1 \to L^1$.
- $A(f)f(x) \to f(x)$ a.e because M maps $L^1 \to L^{1,\infty}$.
- $S_N f(x) \xrightarrow{L^2} f(x)$ because S_N maps $L^2 \to L^2$.
- $S_N f(x) \to f(x)$ a.e because C maps $L^2 \to L^2$, where $C = \sup_N |S_N f(x)|$ is the Calderon maximal operator.

we have not proven the last item, as the proof is nontrivial.

Lecture 14: 10/22

Fix $h > 0$. Let $g_h = \frac{1}{2h}\mathbb{1}_{[-h,h]}$, then let $Av_h : f \mapsto f * g_h$. We have that $Av_h : L^1 \to L^1$ is bounded, meaning that
$$\|Av_h(f)\|_1 \le C\|f\|_1 \qquad f \in L^1(\mathbb{R})$$
In fact, it also holds that for $1 \le p \le \infty$, $Av_h : L^p \to L^p$ is bounded. This is because
$$\begin{aligned} \|Av_h f\|_{L^p} &= \left\| \int_{\mathbb{R}} f(x-y) g_h(y)\, dy \right\|_{L^p} \\ &\le \int_{\mathbb{R}} \|f(x-y)\|_{L^p} g_h(y)\, dy && \text{triangle inequality} \\ &= \|f\|_{L^p} \int_{\mathbb{R}} g_h(y)\, dy \\ &= \|f\|_{L^p} \end{aligned}$$
Note that for $f : \mathbb{R} \to B$, where B is a Banach space, the triangle inequality holds (we have not defined such an integral yet)
$$\left\| \int_{\mathbb{R}} f(x)\, dx \right\|_B \le \int_{\mathbb{R}} \|f(x)\|_B\, dx$$
A natural question is whether the Hardy-Littlewood maximal function also satisfies such properties. Recall that
$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda\, m(|f(x)| > \lambda)$$

This is also the L^1 norm $\sup_\lambda \left\| \lambda \mathbb{1}_{|f|>\lambda} \right\|_1$. For $p < \infty$, we can define the analogous

$$\|f\|_{p,\infty} = \sup_{\lambda>0} \lambda\, m(|f(x)| > \lambda)^{1/p}$$

Which is also the sup of an L^p norm $\sup_\lambda \left\| \lambda \mathbb{1}_{|f|>\lambda} \right\|_p$.

Observation 0.8. The maximal function maps $L^p \to L^{p,\infty}$. For $f \in L^p$,

$$\begin{aligned} Mf(x) &= \sup_{x\in I} \frac{1}{|I|} \int_I |f(x)|\; dx \\ &\leq \sup_{x\in I} \left(\frac{1}{|I|} \int_I |f(z)|^p\; dx \right)^{1/p} \end{aligned}$$

The rest is left as an exercise, following the proof that the maximal function maps L^1 to $L^{1,\infty}$.

Interpolation

Let T be a linear, semilinear, or quasilinear map. A **semilinear** map satisfies

$$\|T(f+g)\| \leq \|T(f)\| + \|T(g)\|$$

A **quasilinear map** satisfies

$$\|T(f+g)\| \leq C\Big(\|T(f)\| + \|T(g)\| \Big)$$

Let, $a, b > 0$, then we say T is of **restricted weak type (a,b)** if there is a constant $C > 0$ such that for all measurable E and for all f such that $|f| \leq \mathbb{1}_E$,

$$\|Tf\|_{b,\infty} \leq C\, m(E)^{1/a}$$

Theorem 19 (Marcinkiewicz)**.** *Let $0 < p_1 < p < p_2 < \infty$, and let T of restricted weak type (p_1, p_1) and (p_2, p_2). Then $T : L^p \to L^p$ boundedly. (One says that T is of strong type (p,p)).*

Proof. We want to show that $\|Tf\|_p \lessapprox \|f\|_p$. Since T is quasilinear,

$$\begin{aligned} |Tf_1 + Tf_2| &\leq C(|Tf_1| + |Tf_2|) \\ |Tf_1 + Tf_2 + Tf_3 + \ldots + Tf_k| &\leq C|Tf_1| + C^2|Tf_2| + \ldots + C^{k-1}|Tf_{k-1}| + C^{k-1}|Tf_k| \end{aligned}$$

We also know that

$$\|Tf\|_p^p \sim \int_0^\infty \alpha^{p-1} m\Big(|Tf| > \alpha\Big)\, d\alpha$$

To make use of the restricted weak type property, we separate (where $y \sim \frac{1}{b^k}$ means $\frac{1}{b^k} < y < \frac{1}{b^{k+1}}$)

$$f = \sum_{k\in\mathbb{Z}} f \mathbb{1}_{|f| \sim \frac{\alpha}{q^k}} =: \sum_{k\in\mathbb{Z}} f_k$$

So we have that

$$\begin{aligned}|Tf| &= \left|T\sum_{k\in\mathbb{Z}} f_k\right| \\ &\le \sum_{k\in\mathbb{Z}} C^{|k|}|Tf_k|\end{aligned}$$

We claim that

$$(\|Tf\| > \alpha) \subseteq \bigcup_{k\in\mathbb{Z}} \{C^{|k|}\|Tf_k\| > \frac{\alpha}{3\cdot 2^{|k|}}\}$$

Thus, we have that

$$m\Big(\|Tf\| > \alpha\Big) \le \sum_{k\in\mathbb{Z}} m\Big(\|Tf_k\| > \frac{\alpha}{3(2C)^{|k|}}\Big)$$

so that

$$\int_0^\infty \alpha^{p-1} m\Big(\|Tf\| > \alpha\Big)\, d\alpha \lessapprox \sum_{k\in\mathbb{Z}} \int_0^\infty \alpha^{p-1} m\Big(\|Tf_k\| > \frac{\alpha}{3(2C)^{|k|}}\Big)\, d\alpha$$

We split this into a sum over $k \ge 0$ and $k < 0$. We estimate each side. For the sum over $k \ge 0$,

$$\sum_{k\in\mathbb{Z}_{\ge 0}} \int_0^\infty \alpha^{p-1} m\Big(\|Tf_k\| > \frac{\alpha}{3(2C)^{|k|}}\Big)\, d\alpha$$

We use the (p_2, p_2) information to see

$$m\Big(\|Tf_k\| > \frac{\alpha}{3\cdot(2C)^k}\Big) \le \Big[\frac{3(2C)^k}{\alpha}\Big]^{p_2}\Big(\frac{\alpha}{q^k}\Big)^{p_2} m\Big(\|Tf\| \sim \frac{\alpha}{q^k}\Big)$$

...

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Continuing the proof the above is equal to

$$\Big[\frac{3(2C)^k}{q^k}\Big]^{p_2} m\Big(\|Tf\| \sim \frac{\alpha}{q^k}\Big)$$

We can sum use this bound to sum the geometric series over $k \ge 0$.

$$\begin{aligned}\sum_{k\ge 0}\Big[\frac{3(2C)^k}{q^k}\Big]^{p_2} m\Big(\|Tf\| \sim \frac{\alpha}{q^k}\Big) &\lessapprox \sum_{k\ge 0}\Big(\int_0^\infty \alpha^{p-1} m\Big(|f| > \alpha/q^k\Big)\, d\alpha\Big)\Big[\frac{3(2C)^k}{q^k}\Big]^{p_2} \\ &= \Big[\frac{3(2C)^k}{q^k}\Big]^{p_2} \cdot \int_0^\infty \beta^{p-1}(q^k)^{p-1} m(|f| > \beta)\, d\beta \cdot q^k \quad \text{change variables} \\ &= \|f\|_p^p \sum_{k=0}^\infty \Big[\frac{3(2C)^k}{q^k}\Big]^{p_2}(q^k)^{p-1}q^k \\ &= \sum_{k=0}^\infty \Big(\Big[\frac{2C}{q}\Big]^{p_2} q^p\Big)^k \\ &= \sum_{k=0}^\infty \Big(\Big[2C\Big]^{p_2} q^{p-p_2}\Big)^k\end{aligned}$$

Thus, if q is chosen large enough, $(2C)^{p_2}q^{p-p_2}$ is less than 1, and the series converges.

Now, we estimate the sum over $k<0$. We use (p_1,p_1) information, to see that

$$m\Big(\|Tf_k\| > \frac{\alpha}{3(2C)^k}\Big) \le \Big[\frac{3(2C)^k}{\alpha}\Big]^{p_1}\Big(\frac{\alpha}{q^k}\Big)^{p_1} m\Big(|f| \sim \frac{\alpha}{q^k}\Big)$$

Then, much like before,

$$\sum_{k<0} \lessapprox \|f\|_p^p \sum_{k<0} r^k$$

where $r = \left[\frac{(2C)}{q}\right]^{p_1} q^p = (2C)^{p_1}q^{p-p_1}$, so that $r>1$ for q large enough (which is what we want since the geometric series is over $k<0$). Hence, the whole series is summable. $\square$

Differentiability of functions

In analogy with the fundamental theorem of calculus, for any x,y we expect

$$F(x)-F(y) = \int_x^y F'(t)\,dt$$

For points $x = x_0 < x_1 < \ldots, x_{n-1} < x_n = y$, we expect

$$|F(x_2)-F(x_1)| + \ldots + |F(x_n)-F(x_{n-1})| \le \int_{\mathbb{R}} \Big|F'(t)\Big|\,dt = \Big\|F'\Big\|_1$$

If $F:[a,b]\to\mathbb{R}$, and $a = t0 < t_1 < \ldots < T_n = b$, the **variation** of F on this partition is

$$\sum_{j=1}^n |F(t_j)-F(t_{j-1})|$$

F is said to have **bounded variation** if there exists an $M>0$ such that all variations are at most M.

Example 0.8.

1. Any F that is real valued, monotone, and bounded is of bounded variation. This is because the variation over $[a,b]$ is simply $F(b)-F(a)$ for any partition. Note that F need not be continuous to have bounded variation.
2. Let F be a function such that F' exists everywhere on $[a,b]$. Then F is of bounded variation. This is due to the mean value theorem, which gives that the variation of any partition is at most $(b-a)\|F'\|_\infty$.

The **total variation** of F on $[a,x]$ is

$$T_F(a,x) = \sup_{\text{partitions}} \sum_{j=1}^n |F(t_j)-F(t_{j-1})|$$

where the supremum is over partitions of $[a,x]$ (this definition also works for F complex valued). If F is real valued, then the **positive variation** is

$$P_F(a,x) = \sup_{\text{partitions}} \sum_{+} F(t_j)-F(t_{j-1})$$

where the sum is over those j such that $F(t_j) - F(t_{j-1}) > 0$. The **negative variation** is defined similarly.

$$N_F(a,x) = \sup_{\text{partitions}} -\sum_{-} F(t_j) - F(t_{j-1})$$

Note the extra negative sign to make N_F a nonnegative quantity.

Lemma 6. *Let F be real valued and with bounded variation on $[a,b]$. Then for any $x \in [a,b]$, one has*

$$\begin{aligned} F(x) - F(a) &= P_F(a,x) - N_F(a,x) \\ T_F(x) &= P_F(a,x) + N_F(a,x) \end{aligned}$$

A proof for the Layman! For continuous functions, one can make an analogy with the miles driven in a car on a trip on a straight line from $F(a)$ to $F(x)$ (where there are generally multiple back and forth movements for some reason). The distance between $F(a)$ and $F(x)$ is $F(x) - F(a)$. The total miles covered during the trip is $T_F(x)$. □

Theorem 20. *A real valued function F on $[a,b]$ is of bounded variation if and only if F is the difference of two increasing bounded functions.*

Proof. If we knew that $F = F_1 - F_2$ in which F_1 and F_2 are increasing and bounded, then the fact that F is of bounded variation is clear.

On the other hand, if F is of bounded variation, then we can apply the lemma to see $F(x) = F(a) + P_F(a,x) - N_F(a,x)$ so that we are done. □

Theorem 21. *Let F be of bounded variation on $[a,b]$. Then F is differentiable* a.e.

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Proof. We write $F = F_1 - F_2$, where F_1 and F_2 are increasing and bounded. First, we will handle the case where F is increasing and continuous.

Lemma 7 (Rising sun lemma)**.** *Let G be real-valued and continuous on $\mathbb{R}$. Let E be the set of all $x \in \mathbb{R}$ such that there exists $h > 0$ such that $G(x+h) > G(x)$.*

- *If E is not empty, it must be open, and so $E = \bigcup_k (a_k, b_k)$ is a disjoint of countably many disjoint intervals.*
- *If (a_k, b_k) is a finite interval, then $G(b_k) = G(a_k)$.*

Proof of lemma. Observe that $G(b_k) \le G(a_k)$ for all k, since $a_k \notin E$. We want to show that $G(b_k) = G(a_k)$. If not, then $G(b_k) < G(a_k)$. The intermediate value theorem implies that there is a $c \in (a_k, b_k)$ with $G(c) = \frac{G(a_k)+G(b_k)}{2}$. Take the sup of all such c, so that c is the largest such number satisfying these properties.

$c \in E$, so we can choose a $d > c$ such that $G(d) > G(c) > G(b_k)$. Since $b_k \notin E$, we know $d < b_k$. Finally, by the intermediate value theorem, we can choose a $c' \in (d, b_k)$ such that $G(c') = G(c)$, contradicting our choice of c. □

Corollary 9. Say $f:[a,b]\to\mathbb{R}$. Then the same is true with the only difference being that if $a_k=a$, then $G(a_k)\le G(b_k)$ (may not necessarily be equal).

Proof. The same proof holds, but the end of the interval $[a,b]$ may cut off a valley $[a_k,b_k)$. $\square$

Now, we define four quantities:

$$\begin{aligned}
D^+(F)(x) &= \limsup_{\substack{h\to 0\\ h>0}} \Delta_h(F)(x)\\
D_+(F)(x) &= \liminf_{\substack{h\to 0\\ h>0}} \Delta_h(F)(x)\\
D^-(F)(x) &= \limsup_{\substack{h\to 0\\ h<0}} \Delta_h(F)(x)\\
D_-(F)(x) &= \liminf_{\substack{h\to 0\\ h<0}} \Delta_h(F)(x)
\end{aligned}$$

Obviously $D_+\le D^+$ and $D_-\le D^-$. It suffices to show that

(a) $D^+F(x)<\infty$ a.e

(b) $D^+F(x)\le D_-F(x)$ a.e

Indeed, if these are true, then we have

$$D^+\le D_-\le D^- \underbrace{\le}_{*} D_+\le D^+<\infty \quad \text{a.e}$$

So that all of the quantities hence the limit as h goes to zero of Δ_h (the derivative of F) exists a.e. To see the inequality $*$, we apply (b) to $-F(-x)$.

Now, fix $\gamma>0$ and define the set $E_\gamma=\{x: D^+F(x)>\gamma\}$. We claim that

$$m(E_\gamma)\le\frac{1}{\gamma}(F(b)-F(a))$$

If this is true, then since

$$\{x: D^+F(x)=\infty\}\subseteq E_\gamma$$

then D^+ is bounded a.e since $m(E_\gamma)\to 0$ and $\gamma\to\infty$.

To prove this claim, consider $G(x)=F(x)-\gamma x$. Note that $\{x: D^+G(x)>0\}$ QQ. Using the lemma of the rising sun, we split $E_\gamma=\bigcup_k(a_k,b_k)$. Thus, we have

$$\begin{aligned}
m(E_\gamma) &\le \sum_k (b_k-a_k)\\
&\le \frac{1}{\gamma}\sum_l F(b_k)-F(a_k) && QQ\\
&= \frac{1}{\gamma}(F(b)-F(a)) && F \text{ increasing}
\end{aligned}$$

so that D^+F is finite a.e.

Now, we want to show $D^+F(x) \leq D^-F(x)$ a.e. For sake on contradiction, suppose that

$$m(D^+F \geq D_-F) > 0$$

Then, for $r, R \in \mathbb{R}_+$, we consider the set

$$E = \{x \in [a,b] \mid D^+F(x) > R,\, r > D_-F(x)\}$$

we will show that $m(E) > 0$ for any $r, R \in \mathbb{R}_+$. Suppose $R > r$. There exists open $\mathcal{O}$ such that $E \subseteq \mathcal{O} \subseteq (a,b)$ with $m(\mathcal{O}) < m(E)\frac{R}{r}$. Then decompose $\mathcal{O} = \bigcup_k I_k$ a union of disjoint open intervals. Fix k in the index set, and consider $G(x) = F(-x) = rx$ on the interval $-I_k$.

The lemma of the rising sun gives $F(b_k) - F(a_k) \leq r(b_k - a_k)$. To see this, let $r > D_-F(x)$. Then there exists $h < 0$ such that $r > \frac{F(x+h)-F(x)}{h}$, so that

$$\begin{aligned} rh &< F(x+h) - F(x) \\ F(x+h) - r(x+h) &> F(x) - rx \\ H(x+h) &> H(x) && \text{where } H(y) := F(y) - ry \end{aligned}$$

This is for $h < 0$. Look instead at $H_2(y) = H(-y)$.

Apply the lemma of the rising sun again and get $(a_{k,j}, b_{k,j}) \subseteq (a_k, b_k)$ for all j such that $F(b_{k,j}) - F(a_{k,j}) \geq R(b_{k,j} - a_{k,j})$. Finally, consider $\mathcal{O}_n = \bigcup_{k,j}(a_{k,j}, b_{k,j})$, and note

$$\begin{aligned} m(\mathcal{O}_n) &\leq \sum_{k,j} \left| b_{k,j} - a_{k,j} \right| \\ &\leq \frac{1}{R} \sum_{k,j} F(b_{k,j}) - F(a_{k,j}) \\ &\leq \frac{1}{R} \sum_k F(b_k) - F(a_k) \\ &\leq \frac{r}{R} \sum_l b_k - a_k \\ &\leq \frac{r}{R} m(I_n) \end{aligned}$$

But $E \subseteq \bigcup_n I_n$, so $E \subseteq \bigcup_n \mathcal{O}_n$ so that

$$\begin{aligned} m(E) &\leq \sum_n m(\mathcal{O}_n) \\ &\leq \frac{r}{R} \sum_n m(I_n) \\ &= \frac{r}{R} m(\mathcal{O}) \\ &< m(E) \end{aligned}$$

Giving our desired contradiction. □

Lecture 17: 10/31

Last time we proved the a.e differentiability of continuous increasing functions of bounded variation. We now want to handle any function of bounded variation. Since any such function can be decomposed into a difference of monotone bounded function, we let $F:[a,b]\to\mathbb{R}$ be just increasing and bounded (not necessarily continuous).

The set of discontinuities of F is countable; let us denote the points of discontinuities $(x_n)_{n\in\mathbb{N}}$ and the jumps $(\alpha_n = F(x_n+) - F(x_n-))_{n\in\mathbb{N}}$. Then we have $F(x_n+) = F(x_n-) + \alpha_n$. Moreover, since F is monotone increasing, we have $F(x_n) = F(x_n-) + \theta_n\alpha_n$ for $\theta_n \in [0,1]$ (meaning $F(x_n)$ can take any value between $F(x_n-)$ and $F(x_n+)$). Define the **normalized jump functions**

$$j_n(x) = \begin{cases} 0 & x < x_n \\ \theta & x = x_n \\ 1 & x > x_n \end{cases}$$

The **grand jump function** is defined

$$J_F(x) = \sum_{n=1}^{\infty} \alpha_n j_n(x)$$

Note the inequalities

$$J_F(x) \le \sum_{n=1}^{\infty} \alpha_n \le F(b) - F(a)$$

Lemma 8.

1. *$J(x)$ is discontinuous exactly at the points x_n and has a jump at x_n equal to that of F (α_n).*
2. *The difference $F(x) - J(x)$ is increasing and continuous.*

Proof.

1. is clear by construction (see hw6 solution)
2. is clear from a picture □

Now, it suffices to prove that $J(x)$ is differentiable a.e.

Theorem 22. *$J'(x)$ is exists and is zero* a.e.

Proof. Since J is increasing, it suffices to show

$$\limsup_{h\to 0} \frac{J(x+h) - J(x)}{h} = 0 \quad \text{a.e}$$

If this is not true, then there is some $\epsilon > 0$ such that the set

$$E = \left\{ x \mid \limsup_{h\to 0} \frac{J(x+h) - J(x)}{h} > \epsilon \right\}$$

has positive measure. We will show that $m(E) = 0$. Pick N large such that $\sum_{n>N} \alpha_n < \eta$. Define

$$J_0(x) = \sum_{n=N+1}^{\infty} \alpha_n j_n(x)$$

This is still a sum of increasing functions, and hence increasing. Then note that

$$0 < J_0(b) - J_0(a) < \eta$$

But then $J - J_0$ is a finite sum of some summands $\alpha_n j_n(x)$, and so the set of points where $\limsup_{h\to 0} \frac{J_0(x+h)-J_0(x)}{h} > \epsilon$ differs from E by at most the points $x_1, \ldots, x_N$.

Let $K \subseteq E$ compact such that $m(K) > m(E)/2 =: \delta/2$ such that $\limsup_{h\to 0} \frac{J_0(x+h)-J_0(x)}{h} > \epsilon$ for all $x \in K$. But then for any $x \in K$, there exists an interval (a_x, b_x) containing x such that

$$J_0(b_x) - J_0(a_x) > \epsilon(b_x - a_x)$$

We choose a finite subcover by compactness, then apply the Vitali covering lemma to get a disjoint subcollection $I_1, \ldots, I_m$ such that $\sum_{j=1}^m m(I_j) \geq m(K)/3$. Finally,

$$\begin{aligned}
\eta &> J_0(b) - J_0(a) \\
&\geq \sum_{j=1}^{m} (J_0(b_j) - J_0(a_j)) \\
&> \epsilon \sum_{j=1}^{m} (b_j - a_j) \\
&= \epsilon \sum_j m(I_j) \\
&> \frac{\epsilon m(K)}{3} \\
&> \frac{\epsilon m(E)}{6}
\end{aligned}$$

since we can choose η independently of ϵ, this shows that $m(E) = 0$, giving our desired contradiction.

□

Observation 0.9. Let F be increasing and continuous. Then F' exists, is measurable, and is Lebesgue integrable, with

$$\int_a^b F'(x)\,dx \leq F(b) - F(a)$$

To see this, consider

$$G_n(x) = \frac{F(x+1/n) - F(x)}{1/n}$$

Then we know that $G_n \to F'(x)$ a.e, and thus F' is measurable since each G_n is. Moreover, $F' \geq 0$. Note that

$$\begin{aligned}
\int_a^b F'(x)\,dx &\leq \liminf \int_a^b G_n(x)\,dx && \text{Fatou} \\
&= \liminf \Big[\frac{1}{1/n}\int_b^{b+1/n} F(x)\,dx - \frac{1}{1/n}\int_a^{a+1/n} F(x)\,dx\Big] \\
&= F(b) - F(a) && \text{continuity}
\end{aligned}$$

Example 0.9. Here we consider the **Cantor-Lebesgue** function, which is a function $F : [0,1] \to [0,1]$ such that F is increasing, $F(0) = 0$, $F(1) = 1$, but $F'(x) = 0$ a.e. Let $\mathcal{C}$ be the standard ternary Cantor set. Then define $F_0(x) = x$, $F_1(x)$ is 0 on $[0,1/3]$, 1/2 on $(1/3,2/3)$,and 1 on $[2/3,1]$. Continue in this way, with F_k being constant on all intervals removed by the kth step of the construction of the Cantor set. Note that

$$|F_{n+1}(x) - F_n(x)| < \frac{1}{2^n}$$

so this is a Cauchy sequence of functions and hence converges to a function F. We call this function the Cantor-Lebesgue function. Then $F'(0) = 0$ on $[0,1] \setminus \mathcal{C}$. Thus, we have

$$\int_a^b F'(x)\, dx < F(1) - F(0) = 1$$

so our above inequality is not in general an equality.

Lecture 18: Absolute Continuity (11/5)

Definition 0.5. A function $F : [a,b] \to \mathbb{C}$ is said to be **absolutely continuous** if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$\sum_{k=0}^{N} |F(b_k) - F(a_k)| < \epsilon \qquad \text{if } \sum_{k=1}^{N} |b_k - a_k| < \delta$$

for any disjoint intervals $(a_k, b_k) \subseteq [a,b]$.

Note that any absolutely continuous has bounded variation and is uniformly continuous. The Cantor-Lebesgue function is not absolutely continuous, as will follow later due to the strict inequality $\int_a^b F'(x)\, dx < F(1) - F(0)$. For absolutely continuous functions, the corresponding relation is actually an equality.

Theorem 23. *If F is absolutely continuous on $[a,b]$, then F' exists* a.e. *If moreover $F'(x) = 0$* a.e*, then F is be constant.*

Proof. We know that F' exists a.e since F has bounded variation. To prove the next part, we use a covering argument.

A collection $\mathcal{B}$ of balls in $\mathbb{R}^d$ is said to be a **Vitali covering** of a measurable set E if for any $x \in E$ and for any $\eta > 0$, there is a $B \in \mathcal{B}$ such that $x \in B$ and $m(B) < \epsilon$.

Lemma 9. *If $m(E) < \infty$ and $\mathcal{B}$ is a Vitali covering of E, then for any $\delta > 0$, there are finitely many disjoint balls $B_1, \ldots, B_N \in \mathcal{B}$ such that*

$$\sum_{k=1}^{N} m(B_k) \geq m(E) - \delta$$

Proof of lemma. Suppose $\delta < m(E)$ (since otherwise we are done). The idea is to apply the Vitali covering lemma that we previously proved. Pick $E' \subseteq E$ compact such that $m(E') > \delta$. Compactness implies that there are finitely many balls from $\mathcal{B}$ which cover E'. Applying the Vitali covering lemma, we have $B_1, \ldots, B_{N_1}$ disjoint such that

$$\sum_{k=1}^{N_1} m(B_k) \geq \gamma m(E') > \gamma\delta$$

where $\gamma = 3^{-d}$. If $\sum_{k=1}^{N_1} m(B_k) \geq m(E) - \delta$, then we are done. If not, then look at

$$E_2 = E \setminus \bigcup_{k=1}^{N_1} \overline{B}_k$$

This means that

$$m(E_2) = m(E) - \sum_{k=1}^{N_1} \overline{B}_k > \delta$$

Then repeat the previous step. We take a E_2' compact with $m(E_2') > \delta$. Since $\mathcal{B}$ is a Vitali covering, it also covers E_2' with balls that do not touch the $\bigcup_{k=1}^{N_1} \overline{B}_k$. Thus, we can take a finite cover of E_2' with balls that are disjoint from the $\bigcup_{k=1}^{N_1} \overline{B}_K$, that satisfies

$$\sum_{k=N_1+1}^{N_2} m(B_k) \geq \gamma\delta$$

Now, we have the our total cover so far satisfies

$$\sum_{k=1}^{N_2} m(B_k) > 2\gamma\delta$$

continue the process as needed. It takes only finitely many steps to terminate, so we will end with a finite disjoint cover. □

Corollary 10. One can arrange these balls from the lemma to satisfy

$$m\Big(E \setminus \bigcup_{k=1}^{N} B_k\Big) < 2\delta$$

Proof of corollary. Let $\mathcal{O}$ open with $E \subseteq \mathcal{O}$ and $m(\mathcal{O} \setminus E) < \delta$. We can assume without loss of generality that all balls in $\mathcal{B}$ are contained in $\mathcal{O}$. Then, letting $B_1, \ldots, B_N$ be a finite disjoint cover from application of the lemma,

$$\Big(E \setminus \bigcup_{k=1}^{N} B_k\Big) \cup \Big(\bigcup_{k=1}^{N} B_k\Big) \subseteq \mathcal{O}$$

Hence, we have that

$$\begin{aligned} m\Big(E \setminus \bigcup_k B_k\Big) &\leq m(\mathcal{O}) - m\Big(\bigcup_k B_k\Big) \\ &\leq m(E) + \delta - (m(E) - \delta) \\ &= 2\delta \end{aligned}$$

□

We return to the proof of the theorem. Let $F' = 0$ a.e. It suffices to prove that $F(b) = F(a)$ for any a, b (since we can shrink the domain and consider any pair of points in this way). Let $E = \{x \in (a,b) : F'(x) = 0\}$. For $x \in E$,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = 0$$

Let $\epsilon > 0$. Then for $x \in E$, there exists an $I_x = (a_x, b_x) \subseteq (a,b)$ with $x \in I_x$ such that

$$|F(b_x) - F(a_x)| < \epsilon |b_x - a_x|$$

and $|b_x - a_x| < \epsilon$. By the lemma, there are finitely many disjoint intervals $I_1, \ldots, I_N$ such that

$$\sum_{j=1}^{N} m(I_j) \geq m(E) - \delta = b - a - \delta$$

Denote the complement

$$E \setminus \bigcup_k I_k = \bigcup_{k=1}^{M} [\alpha_k, \beta_k]$$

Then we have $\sum_{k=1}^{M} |\beta_k - \alpha_k| \leq \delta$, so if we go back and choose a small enough δ, absolute continuity gives that

$$\sum_{k=1}^{M} |F(b_k) - F(a_k)| < \epsilon$$

This gives in total that

$$\begin{aligned} |F(b) - F(a)| &\leq \sum_{k=1}^{N} |F(b_k) - F(a_k)| + \sum_{j=1}^{M} |F(\beta_j) - F(\alpha_j)| \\ &\leq \epsilon(b-a) + \epsilon \end{aligned}$$

□

Theorem 24. *If F is absolutely continuous on $[a,b]$, then*

$$\int_a^x F'(y)\, dy = F(x) - F(a) \qquad \forall x \in [a,b]$$

If f is Lebesgue integrable on $[a,b]$, then the function

$$F(x) = \int_a^x f(y)\, dy$$

is absolutely continuous and $F'(x) = f(x)$ a.e.

Proof. Consider $G(x) = \int_a^x F'(y)\, dy$. Observe that $(F-G)' = 0$ a.e. Thus, $F - G = C$ is constant, so $F(a) - G(a) = F(a) = C$, and thus $F(x) - G(x) = F(a)$ means that $F(x) - F(a) = G(x)$ a.e. □

Lecture 19: Rectifiable Curves, Isoperimetric Inequality (11/7)

Let $F : [a,b] \to \mathbb{C} \cong \mathbb{R}^2$, $F(t) = (x(t), y(t))$. Also, assume F continuous. Then we consider the variation

$$L = \sup_{a = t_0 < \ldots < t_N = b} \sum_{i=1}^{N} |F(t_i) - F(t_{i-1})|$$

If L is finite, we say that the curve is **rectifiable**. A natural question to ask for curves is when does the arclength formula hold

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Example 0.10. Let C be the Cantor-Lebesgue function and consider the curve $F(t) = (C(t), C(t))$. Note that $F(0) = (0,0)$ and $F(1) = (1,1)$. The arclength formula does not hold here, since the derivative of the components are $C'(t) = 0$ a.e.

Theorem 25. *If both $x(t)$ and $y(t)$ are absolutely continuous, then*

$$L = \int_a^b \left|F'(t)\right| \, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, dt = T_F(a,b)$$

Proof. Let $a = t_0 < \ldots < t_N = b$ be a partition, and consider

$$\begin{aligned} \sum_{j=1}^N \left|F(t_j) - F(t_{j-1})\right| &= \sum_{j=1}^N \left| \int_{t_{j-1}}^{t_j} F'(s) \, ds \right| && \text{absolute continuity} \\ &\leq \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \left|F'(s)\right| \, ds \end{aligned}$$

For the other inequality, let $\epsilon > 0$. Let $F' = g + h$, where g is a step function and $\|h\|_1 < \epsilon$. We can do this by density of step functions in L^1. Then $F = G + H$, where

$$G(x) = \int_a^x g(t) \, dt \qquad H(x) = \int_a^x h(t) \, dt$$

Then we have that

$$\begin{aligned} T_H(a,b) &\leq \int_a^b \left|H'(t)\right| \, dt = \int_a^b |h(t)| \, dt < \epsilon \\ T_F(a,b) &\geq T_G(a,b) - T_H(a,b) \\ T_F(a,b) &\geq T_G(a,b) - \epsilon \end{aligned}$$

Consider the partition associated to the steps of g, so g is constant within each interval. Then

$$\begin{aligned} T_G(a,b) &\geq \sum_j \left|G(t_j) - G(t_{j-1})\right| \\ &= \sum_j \left| \int_{t_{j-1}}^{t_j} g(t) \, dt \right| \\ &= \sum_j \int_{t_{j-1}}^{t_j} |g(t)| \, dt && \text{choice of partition} \\ &= \int_a^b |g(t)| \, dt \\ &\geq \int_a^b \left|F'(t)\right| \, dt - \epsilon \end{aligned}$$

so that in total,

$$T_F(a,b) \geq \int_a^b \left|F'(t)\right| \, dt - 2\epsilon$$

□

Minkowski Content

Our goal is to prove that an isometric inequality—if $\Omega \subseteq \mathbb{R}^2$ is a bounded open set such that $\overline{\Omega} \setminus \Omega$ is a rectifiable curve Γ, then $4\pi m(\Omega) \leq L(\Gamma)^2$

A curve $z(t) = (x(t), y(t))$, $t \in [a,b]$, is **simple** if it is injective. It is called **quasi-simple** if it is injective except on finitely many points.

Let $K \subseteq \mathbb{C}$ compact and $\delta > 0$. Define

$$K^\delta = \{x \in \mathbb{R}^2 \mid d(x,K) < \delta\}$$

We say K has Minkowski content if

$$\lim_{\delta \to 0} \frac{m(K^\delta)}{2\delta} =: M(K) \quad \text{exists}$$

Example 0.11. A line segment has Minkowski content. A ball does not.

Theorem 26. *Let $\Gamma = \{z(t) \mid t \in [a,b]\}$ is a quasi-simple curve. Then the Minkowski content of Γ exists if and only if Γ is rectifiable. Moreover,*

$$L = M(\Gamma)$$

Proof. Define the quantities

$$M^*(k) = \limsup_{\delta \to 0} \frac{m(K^\delta)}{2\delta}$$
$$M_*(k) = \liminf_{\delta \to 0} \frac{m(K^\delta)}{2\delta}$$

It suffices to prove the following two propositions

Proposition 14. If Γ is quasi-simple and $M_*(\Gamma) < \infty$, then Γ is rectifiable and its length L is bounded as

$$L(\Gamma) \leq M_*(\Gamma)$$

Proof.

Lemma 10. *Let $\Gamma = \{z(t) \mid t \in [a,b]\}$ and let $\Delta = |z(b) - z(a)|$. Then*

$$m(\Gamma^\delta) \geq 2\delta\Delta$$

Proof.

$$\begin{aligned} m(\Gamma^\delta) &\geq \int_A^B m_{\mathbb{R}}((\Gamma^\delta)_x)\, dx \\ &\geq 2\delta(B - A) \\ &= 2\delta\Delta \end{aligned}$$

□

Now, consider

$$L_P = \sum_{j=1}^{N} \left| z(t_j) - z(t_{j-1}) \right|$$

There exists disjoint $I_j = [a_j, b_j] \subset (t_{j-1}, t_j)$ such that

$$\sum_{j=1}^{N} \left| z(b_j) - z(a_j) \right| \geq L_p - \epsilon$$

Let $\Gamma_j = \Gamma \mid_{[a_j,b_j]}$. Then $\Gamma^\delta \supset \Gamma_1^\delta \cup \ldots \cup \Gamma_N^\delta$ where Γ_j are all disjoint (by choice of small δ). Then we have that

$$\begin{aligned} m(\Gamma^\delta) &\geq \sum_j m(\Gamma_j^\delta) \\ &\geq \sum_j 2\delta \left| z(b_j) - z(a_j) \right| && \text{lemma} \\ &\geq 2\delta(L_p - \epsilon) \end{aligned}$$

□

Proposition 15. If Γ is rectifiable, then

$$M^*(\Gamma) \leq L(\Gamma)$$

Proof. Assume that $z(t)$ is the arclength parameterization, so $|z'(t)| = 1$. We want

$$M^*(\Gamma) \leq L(\Gamma)$$

Fix $\epsilon > 0$. We claim that there exists a set $E_\epsilon \subseteq \mathbb{R}$ and $r_\epsilon > 0$ such that $m(E_\epsilon) < \epsilon$ and

$$\sup_{0<|h|<r_\epsilon} \left| \frac{z(s+h) - z(s)}{h} - z'(s) \right| < \epsilon \qquad \forall s \in [0, L] \setminus E_\epsilon \qquad (*)$$

To see this, consider the functions

$$F_n(s) = \sup_{|h|<1/n} \left| \frac{z(s+h) - z(s)}{h} - z'(s) \right|$$

Note that $F_n(s) \to 0$ a.e. Then Egorov gives that there exists E_ϵ with $m(E_\epsilon) < \epsilon$ such that $F_n(s) \to 0$ uniformly for in $[0, L] \setminus E_\epsilon$.

Fix $\rho < r_\epsilon < 1$, and partition $[0, L]$ into intervals $I_1, \ldots, I_N$, where $m(I_j) = \rho$ (except for possibly the last interval) and $N \leq \frac{L}{\rho} + 1$. We call I_j good if $I_j \subsetneq E_\epsilon$, and bad if $I_j \subseteq E_\epsilon$. Denote $\Gamma_j = \Gamma \mid_{I_j}$. Then we have

$$m(\Gamma^\delta) \leq \sum_{j=1}^{N} m(\Gamma_j^\delta)$$

We estimate this term on the right with cases

- If I_j is good, then there exists an $s_0 \in I_j$ such that $(*)$ holds that S_0, so

$$\sup_{0<|h|<r_\epsilon} \left| \frac{z(s_0+h)-z(s_0)}{h} - z'(s_0) \right| < \epsilon$$

  without loss of generality, $z(s_0)=0$ and $z'(s_0)=1$. Say $I_j=[a_j,b_j]$. We want $s_0+h \in [a_j,b_j]$, so $h \in [a_j-s_0, b_j-s_0] \subseteq [-\delta,\delta]$. Since $\delta < r_\epsilon$, we know

$$\begin{aligned} \left|z(s_0+h)-h\right| &< \epsilon|h| \\ \left|z(s_0+h)-h\right| &< \rho && h \in [a_j-s_0, b_j-s_0] \end{aligned}$$

  Then we have that

$$\begin{aligned} \Gamma_j &\subseteq [a_j-s_0-\epsilon\rho, b_j-s_0+\epsilon\rho] \times [-\epsilon\rho,\epsilon\rho] \\ \Gamma_j^\delta &\subseteq [a_j-s_0-\epsilon\rho-\delta, b_j-s_0+\epsilon\rho+\delta] \times [-\epsilon\rho-\delta, \epsilon\rho+\delta] \\ m(\Gamma_j^\delta) &\leq (\rho+2\epsilon\rho+2\delta)(2\epsilon\rho+2\delta) \\ &\leq 2\delta\rho + O(\epsilon\delta\rho+\delta^2+\epsilon\rho^2) \end{aligned}$$

- If I_j is bad, i.e. $I_j \subseteq E_\epsilon$, then we have

$$\begin{aligned} \left|z(s_1)-z(s_2)\right| &\leq |s_1-s_2| \\ &\leq \rho && s_1,s_2 \in I_j \subseteq E_\epsilon \end{aligned}$$

  Ths means that Γ_j is contained within a ball of radius ρ, so that

$$m(\Gamma_j^\delta) = O(\rho^2+\delta^2)$$

Thus, our estimates give

$$m(\Gamma^\delta) \leq 2\delta L + 2\delta\rho + O(\epsilon\delta+\delta^2/\rho+\epsilon\rho) + O((\epsilon/\rho+1)(\delta^2+\rho^2))$$

where we have used that $N \leq L/\rho+1$. Thus, ,we have

$$\begin{aligned} \frac{m(\Gamma^\delta)}{2\delta} &\leq L+\rho+O(\epsilon+\delta/\rho+\epsilon\rho/\delta) + O((\epsilon/\rho+1)(\delta^2+\rho^2)/\delta) \\ &= L + O\Big(\rho+\epsilon+\frac{\delta}{\rho}+\frac{\epsilon\rho}{\delta}+\frac{\rho^2}{\delta}+\delta\Big) \end{aligned}$$

Now, take $\rho = \frac{\delta}{\epsilon^{1/2}}$. Then for any δ we have

$$\frac{m(\Gamma^\delta)}{2\delta} < L + O\Big(\frac{\delta}{\epsilon^{1/2}}+\epsilon+\epsilon^{1/2}+\frac{\delta}{\epsilon}\Big)$$

So that taking $\limsup$ as $\delta \to 0$, we have

$$M^*(\Gamma) \leq L + O(\epsilon+\epsilon^{1/2})$$

And we are done. $\square$

$\square$

Lecture 20: Isoperimetric Inequality (11/12)

First, we finished the proof of Proposition 15.

Theorem 27 (Isoperimetric Inequality)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open subset of the plane such that $\overline{\Omega} \setminus \Omega$ is a rectifiable curve Γ, with length $l(\Gamma)$. Then*

$$4\pi m(\Omega) \leq l(\Gamma)^2$$

Proof. Let $\epsilon > 0$. Consider the sets

$$\begin{aligned}\Omega_+(\delta) &= \{x \in \mathbb{R}^2 \mid d(x, \overline{\Omega}) < \delta\} \\ \Omega_-(\delta) &= \{x \in \mathbb{R}^2 \mid d(x, \Omega^c) \geq \delta\}\end{aligned}$$

Note that $\Omega_-(\delta) \subseteq \Omega \subseteq \Omega_+(\delta)$ and

$$\Omega_+(\delta) = \Omega_-(\delta) \cup \Gamma^\delta$$

Let $D(\delta) = B_\delta(0)$. Then we have the relations

$$\begin{aligned}\Omega_+(\delta) &\supseteq \Omega + D(\delta) \\ \Omega &\supseteq \Omega_- + D(\delta)\end{aligned}$$

This means that the measures satisfy

$$\begin{aligned}m(\Omega_+(\delta)) &\geq m(\Omega + D(\delta)) \\ &\geq \left[m(\Omega)^{1/2} + m(D(\delta))^{1/2}\right]^2 && \text{Brunn-Minkowski} \\ &= \left[m(\Omega)^{1/2} + \pi^{1/2}\delta\right]^2 \\ &\geq m(\Omega) + 2m(\Omega)^{1/2}\pi^{1/2}\delta\end{aligned}$$

Likewise,

$$\begin{aligned}m(\Omega) &\geq m(\Omega_-(\delta)) + 2\pi^{1/2}\delta m(\Omega_-(\delta))^{1/2} \\ -m(\Omega_-(\delta)) &\geq -m(\Omega) + 2\pi^{1/2}\delta m(\Omega_-(\delta))^{1/2}\end{aligned}$$

So that we have the big bound

$$\begin{aligned}&m(\Gamma^\delta) \geq 2\pi^{1/2}\delta\left[m(\Omega)^{1/2} + m(\Omega_-(\delta))^{1/2}\right] \\ &\frac{m(\Gamma^\delta)}{2\delta} \geq \pi^{1/2}\left[m(\Omega)^{1/2} + m(\Omega_-(\delta))^{1/2}\right] \\ &M^*(\Gamma) \geq \pi^{1/2}[2m(\Omega)^{1/2}] && \text{taking } \delta \to 0\end{aligned}$$

Thus, since Γ is rectifiable, we are done. $\square$

Lecture 21: Abstract Measure and Integration (11/14)

Definition 0.6. A **measurable space** is a set X together with a **sigma-algebra** $\mathcal{M}$ of measurable sets, which is a non-empty collections of subsets of X closed under complements, countable unions, and countable intersections.

A **measure** is a function $\mu : \mathcal{M} \to [0,\infty]$ such that

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$$

for any collection of disjoint sets $(E_k)_{k\in\mathbb{N}}$.

Such a triple $(X,\mathcal{M},\mu)$ is called a **measure space**. A measure space is called $\sigma-$finite if there are sets $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that $X = \bigcup_n X_n$ and $\mu(X_n) < \infty$ for each $n \in \mathbb{N}$.

Example 0.12. The counting measure, when $X = \mathbb{N}$. The $\sigma-$algebra is the power set $2^{\mathbb{N}}$. The measure of a set is the number of elements if it is finite, and ∞ otherwise.

Example 0.13.

$$\mu(A) = \int_A |f(x)| \ dx$$

For $f \in L^1(\mathbb{R}^d)$, where the σ-algebra is the Lebesgue σ-algebra.

Our goal is to construct measures in abstract general contexts. Let X be a set.

Definition 0.7. An **exterior measure** (or **outer measure**) on X is a function $\mu_* : 2^X \to [0,\infty]$ such that

1. $\mu_*(\emptyset) = 0$
2. If $E_1 \subseteq E_2$, then $\mu_*(E_1) \leq \mu_*(E_2)$
3. For any E_k, we have
$$\mu_*\Big(\bigcup_{k\in\mathbb{N}} E_k\Big) \leq \sum_{k\in\mathbb{N}} \mu_*(E_k)$$

Our idea is to let E be measurable if and only if for all $A \subseteq X$ one has

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c)$$

The $\leq$ inequality is always free. Note in particular that $\mu_*(E) = 0$ for free. The collection of measurable sets under this rule are the Caratheodory measurable sets.

Theorem 28. *For every exterior measure μ_* on X, the collection $\mathcal{M}$ of Caratheodory measurable sets form a σ-algebra. Moreover, $\mu_* |_{\mathcal{M}}$ is a measure.*

Proof. Note that $\emptyset$ and $X \in \mathcal{M}$. Moreover, if $E \in \mathcal{M}$ the clearly E^c is as well.

Now we show that $\mathcal{M}$ is closed under finite unions. Let E_1, $E_2 \in \mathcal{M}$ and $A \subseteq X$. We have that

$$\begin{aligned}
\mu_*(A) &= \mu_*(E_2 \cap A) + \mu_*(E_2^c \cap A) \\
&= \mu_*(E_1 \cap E_2 \cap A) + \mu_*(E_1^c \cap E_2 \cap A) + \mu_*(E_1 \cap E_2^c \cap A) + \mu_*(E_1^c \cap E_2^c \cap A) \\
&\geq \mu_*((E_1 \cup E_2) \cap A) + \mu_*(E_1^c \cap E_2^c \cap A) \\
&= \mu_*((E_1 \cup E_2) \cap A) + \mu_*((E_1 \cup E_2)^c \cap A)
\end{aligned}$$

where we use that $A \cup B = (A \cap B) \cup (A^c \cap B) \cup (A \cap B^2)$.

Assume that $E_1 \cap E_2 = \emptyset$. Then we have that

$$\begin{aligned} \mu_*(E_1 \cup E_2) &= \mu_*(E_1 \cap (E_1 \cup E_2)) + \mu_*(E_1^c \cap (E_1 \cup E_2)) \\ &= \mu_*(E_1) + \mu_*(E_2) \end{aligned}$$

Now, we show that $\mathcal{M}$ is closed under countable unions. For $(A_n)_n \subseteq \mathcal{M}$, note that we can work with disjoint sets. This is because

$$\bigcup_n = A_1 \cup [A_2 \setminus A_1] \cup [A_3 \setminus (A_1 \cup A_2)] \dots$$

And we have just shown that $\mathcal{M}$ is closed under complements and finite unions, hence finite intersections, so each of the unioned sets is measurable.

So let E_n be disjoint sets in $\mathcal{M}$. Define $G_n = \bigcup_{j=1}^n E_j$ and $G = \bigcup_j E_j$. Note that $G_n \in \mathcal{M}$, so for any $A \subseteq X$ we have

$$\begin{aligned} \mu_*(G_n \cap A) &= \mu_*(E_n \cap (G_n \cap A)) + \mu_*(E_n^c \cap (G_n \cap A)) \\ &= \mu_*(E_n \cap A) + \mu_*(G_{n-1} \cap A) \\ &= \sum_{j=1}^n \mu_*(E_j \cap A) && \text{by induction} \end{aligned}$$

Observe that $G^c \subseteq G_n$, so we have

$$\begin{aligned} \mu_*(A) &= \mu_*(G_n \cap A) + \mu_*(G_n^c \cap A) \\ &\geq \Big[\sum_{j=1}^n \mu_*(E_j \cap A)\Big] + \mu_*(G^c \cap A) \end{aligned}$$

This holds for any n, so we have

$$\begin{aligned} \mu_*(A) &\geq \Big[\sum_{j=1}^\infty \mu_*(E_j \cap A)\Big] + \mu_*(G^c \cap A) \\ &\geq \mu_*(G \cap A) + \mu_*(G^c \cap A) \end{aligned}$$

so $G \in \mathcal{M}$. Note lastly that $\mu_*(G \cap A) + \mu_*(G^c \cap A) \geq \mu_*(A)$. Then applying this and the chain of inequalities above with $A = G$, we have that

$$\mu_*(G) = \sum_{j=1}^\infty \mu_*(E_j \cap G) + \mu_*(G^c \cap G) = \sum_{j=1}^\infty \mu_*(E_j)$$

so that μ_* is indeed a measure on $\mathcal{M}$. □

Moreover, any $(X, \mathcal{M}, \mu)$ constructed as by the theorem is complete, meaning that if $F \subseteq \mathcal{M}$ with $\mu(F) = 0$, it holds that any $A \subseteq F$ is measurable $A \in \mathcal{M}$.

Metric exterior measures

We move on to measures with more structure. Say (X,d) is a metric space. An exterior measure μ_* is said to be a **metric exterior measure** if $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$ for any A, $B \subseteq X$ such that $d(A,B) > 0$.

Theorem 29. *If μ_* is a metric exterior measure on a metric space (X,d), then the Borel sets B_X in X are Caratheodory measurable. Thus, $\mu_* |_{B_X}$ is a measure.*

Proof. We will show that any closed $F \subseteq X$ is Caratheodory measurable. Thus, we must show that for any $A \subseteq X$, $\mu_*(A) \geq \mu_*(F \cap A) + \mu_*(F^c \cap A)$. Note that we can assume $\mu_*(A) < \infty$. Define $A_n = \{x \in F^c \cap A \mid d(x,F) \geq \frac{1}{n}\}$. Then $A_n \subseteq A_{n+1}$ and $F^c \cap A = \bigcup_{n \in \mathbb{Z}_+} A_n$ (because F is closed!). Also, we have

$$d(F \cap A,\ A_n) \geq \frac{1}{n}$$

so we can use the metric property

$$\begin{aligned} \mu_*(A) &= \mu_*((F \cap A) \cup A_n) \\ &= \mu_*(F \cap A) + \mu_*(A_n) \end{aligned}$$

It suffices to show that $\mu_*(A_n) \to \mu_*(F^c \cap A)$.

——

Define $B_n = A_{n+1} \setminus A_n$. Then

$$d(B_{n+1}, A_n) \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

Thus,

$$\begin{aligned} \mu_*(A_{2k+1}) &\geq \mu_*(B_{2k} \cup A_{2k-1}) \\ &= \mu_*(B_{2k}) + \mu_*(A_{2k-1}) \\ &\geq \sum_{j=1}^{k} \mu_*(B_{2j}) \end{aligned}$$

and similarly,

$$\mu_*(A_{2k}) \geq \sum_{j=1}^{k} \mu_*(B_{2j-1})$$

Observe that both $\sum_j \mu_*(B_{2j})$ and $\sum_j \mu_*(B_{2j-1})$ are both convergent, since they are both bounded by $\mu_*(A)$. Finally, we have (noting that $F^c \cap A = A_n \cup \bigcup_{j=n+1}^{\infty} B_j$).

$$\mu_*(A_n) \leq \mu_*(F^c \cap A) \leq \mu_*(A_n) + \sum_{j=n+1}^{\infty} \mu_*(B_j)$$

and note that the tails $\sum_{j=n+1}^{\infty} \mu_*(B_j) \to 0$ by convergence of the whole series, so we are done. $\square$

Lecture 22: (11/19)

First, we finished the proof of Theorem 29.

We saw that we can turn a set X into a measure space if we

1. Start with $\mu_* : 2^X \to [0, \infty]$ an exterior measure
2. Let the Caratheodory measurable sets $\mathcal{M}$ be the σ-algebra of the space and restrict μ_* to $\mathcal{M}$

Moreover, if X is a metric space, then starting with a metric exterior measure μ_*, then we can take the σ-algebra to be the Borel σ-algebra and μ_* restricted to this σ-algebra is a measure.

Proposition 16. Let μ be a Borel measure which is finite on all balls of finite radius. Then for any E Borel and for any $\epsilon > 0$, there exists $\mathcal{O}$ open and F closed such that $F \subseteq E \subseteq \mathcal{O}$ and

$$\mu(\mathcal{O} \setminus E) < \epsilon, \qquad \mu(E \setminus F) < \epsilon$$

Proof. Note that by taking complements, we need only prove existence of such an F.

First case: Assume $E = F^* = \bigcup_{k=1}^{\infty} F_k$, F_k are all closed. We wish to show that for all $\epsilon > 0$, there is an F closed with $F \subseteq F^*$ such that $\mu(F^* \setminus F) < \epsilon$. Assume without loss of generality $F_k \nearrow$. Fix $x_0 \in X$ and $B_n = \{x \mid d(x, x_0) < n\}$ open balls, with $B_0 = \emptyset$. Clearly, $\bigcup_{n=0}^{\infty} B_n = X$ and $F^* = \bigcup_n F^* \cap (\overline{B}_n \setminus B_{n-1})$. We have that

$$\mu(F_k \cap (\overline{B}_n \setminus B_{n-1})) \to \mu(F^* \cap (\overline{B}_n \setminus B_{n-1}))$$

For all n, there is an $N(n)$ such that

$$\mu((F^* \setminus F_{N(n)}) \cap (\overline{B}_n \setminus B_{n-1})) < \frac{\epsilon}{2^n}$$

We can define $F = \bigcup_{n=1}^{\infty} F_{N(n)} \cap (\overline{B}_n \setminus B_{n-1})$, and note that the error is

$$(F^* \setminus F) \subseteq \bigcup_n ((F^* \setminus F_{N(n)}) \cap (\overline{B}_n \setminus B_{n-1}))$$

and is hence bounded by ϵ as seen by summing the geometric series. We just need to prove F closed.

General case: Let $\mathcal{C}$ be the collection of all Borel sets for which the property is true. Note that for any $A \in \mathcal{C}$, $A^c \in \mathcal{C}$. Alos, if $E_k \in \mathcal{C}$ for $k \in \mathbb{N}$, then $\bigcup_k E_k \in \mathcal{C}$. To see this, for all k we can choose $E_k \subseteq \mathcal{O}_k$ and $\mu(\mathcal{O}_k \setminus E_k) < \frac{\epsilon}{2^k}$. Let $\mathcal{O} = \bigcup_k \mathcal{O}_k$, and note that $E \subseteq \mathcal{O}$ and

$$\mu(\mathcal{O} \setminus E) \leq \sum_k \mu(\mathcal{O}_k \setminus E_k) < \epsilon$$

Now take $F_k \subseteq E_k$ closed with $\mu(E_k \setminus F_k) < \frac{\epsilon}{2^k}$. Let $F^* = \bigcup_k F_k$. Note that $\mu(E \setminus F^*) < \epsilon$. However, F^* is not closed in general, so we use case 1. And thus know that $\mathcal{C}$ is a σ-algebra.

We show that every open set is in $\mathcal{C}$. We take $\mathcal{O}$ open, and wish to find F closed with $F \subseteq \mathcal{O}$ and with $\mu(\mathcal{O} \setminus F) < \epsilon$. Let $F_k = \{x \in \overline{B}_k \mid d(x, \mathcal{O}^c) \geq 1/k\}$. We are done since $\mathcal{O} = \bigcup_k F_k$. Hence, the Borel sets are all contained within $\mathcal{C}$. $\square$

We move on to other ways of making sets into measure spaces. Let X be a set, and A an **algebra** of subsets of it, meaning a collection that is closed under complements, finite unions, and finite intersections. A **premeasure** on A is a function $\mu_0 : A \to [0,\infty]$ such that

- $\mu_0(\emptyset) = 0$
- If $(E_n)_{n\in\mathbb{N}}$ is a countable collection of disjoint sets with $E_n \in A$ and $\bigcup_n E_n \in A$, then

$$\mu_0\Big(\bigcup_n E_n\Big) = \sum_n \mu_0(E_n)$$

Lecture 23: Carathéodory Extension (11/21)

Example 0.14. The set of all finite unions of rectangles in $\mathbb{R}^d$ forms an algebra.

Lemma 11. *Let μ_0 be a premeasure on an algebra $\mathcal{A}$. Define for all $E \subseteq X$*

$$\mu_*(E) := \inf\Big\{\sum_{j=1}^{\infty} \mu_0(E_j) \mid E \subseteq \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{A}\,\forall j\Big\}$$

Then μ_* is an exterior measure on X such that

(1) $\mu_*(E) = \mu_0(E)$ for all $E \in \mathcal{A}$

(2) All sets in $\mathcal{A}$ are Carathéodory measurable under this exterior measure

Proof. The proof that μ_* is an exterior measure is analogous to the proofs of properties of the Lebesgue outer measure.

Let $E \in \mathcal{A}$. Observe that $\mu_*(E) \le \mu_0(E)$ from definition. Now, let $E \subseteq \bigcup_{j\in\mathbb{N}} E_j$, $E_j \in \mathcal{A}$. Consider $E_1, E_2 \setminus E_1, E_3 \setminus (E_1 \cup E_2)$. Intersect them all with E, and call them F_j. Note $\bigcup_j F_j = E$, so

$$\begin{aligned}\mu_0(E) &= \sum_{j\in\mathbb{N}} \mu_0(F_j)\\ &\le \sum_{j\in\mathbb{N}} \mu_0(E_j)\end{aligned}$$

so by taking the inf, $\mu_0(E) \le \mu_*(E)$.

Now, we show that any $E \in \mathcal{A}$ is Carathéodory. Let $B \subseteq X$. We wish to show that $\mu_*(B) \ge \mu_*(E \cap B) + \mu_*(E^c \cap B)$. Let $\epsilon > 0$. Then for some covering, we have $\mu_*(B) \le \sum_{j\in\mathbb{N}} \mu_0(E_j) \le \mu_*(B) + \epsilon$. Note that

$$\begin{aligned}\mu_*(B) + \epsilon &\ge \sum_{j\in\mathbb{N}} \mu_0(E_j)\\ &= \sum_{j\in\mathbb{N}} \mu_0(E \cap E_j) + \sum_{j\in\mathbb{N}} \mu_0(E^c \cap E_j)\\ &\ge \mu_*(E \cap B) + \mu_*(E^c \cap B)\end{aligned}$$

so we are done. ∎

Theorem 30 (Carathéodory Extension)**.** *Suppose $\mathcal{A}$ is an algebra of sets in X, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ that extends μ_0. If we ask μ to be σ-finite, then there is a unique extension.*

Proof. By the previous lemma, μ_0 extends to μ_* on $\mathcal{C}$, the Carathéodory sets. Moreover, $\mathcal{M} \subseteq \mathcal{C}$. Thus, we define $\mu = \mu_* |_{\mathcal{M}}$, which is a measure on $\mathcal{M}$ that extends μ_0.

We now prove uniqueness. Let ν be a measure on $\mathcal{M}$ such that $\mu |_{\mathcal{A}} = \nu |_{\mathcal{A}}$. We wish to show that $\mu = \nu$ on $\mathcal{M}$. Let $F \in \mathcal{M}$. First, assume $\mu(F) < \infty$.

Let $F \subseteq \bigcup_j E_j$, $E_j \in \mathcal{A}$ for all j.

$$\begin{aligned} \nu(F) &\leq \sum_j \nu(E_j) \\ &= \sum_j \mu_0(E_j) \\ &= \mu(E) \end{aligned}$$

Now, assume $E = \bigcup_j E_j$. Then we have

$$\begin{aligned} \nu(E) &= \lim_{n\to\infty} \nu(\bigcup_{j=1}^n E_j) \\ &= \lim_{n\to\infty} \mu(\bigcup_{j=1}^n E_j) && \text{finite union, stays in } \mathcal{A} \\ &= \mu(F) \end{aligned}$$

Now choose E_j such that $\mu(E) \leq \mu(F) + \epsilon$, so $\mu(E \setminus F) < \epsilon$. This means that

$$\begin{aligned} \mu(F) &\leq \mu(E) \\ &= \nu(E) \\ &= \nu(F) + \nu(E \setminus F) \\ &\leq \mu(E \setminus F) \\ &< \epsilon \end{aligned}$$

If $\mu(F) = \infty$, then since μ is σ-finite, there is E_j of finite measure all disjoint such that $\bigcup_j E_J = X$. Then for any $F \in \mathcal{M}$,

$$\begin{aligned} \mu(F) &= \sum_j \mu(F \cap E_j) \\ &= \sum_j \nu(F \cap E_j) \\ &= \nu(F) \end{aligned}$$

□

Integration on a measure space

We take $(X, \mathcal{M}, \mu)$ a σ-finite measure space, generally assumed to be complete. We follow an analogous development of integration as when our measure space was $\mathbb{R}^d$ with Lebesgue measure.

- A function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is **measurable** if $f^{-1}([-\infty, a)) \in \mathcal{M}$.
- For f, g measurable, we say $f = g$ a.e if $\mu(f \neq g) = 0$.
- Simple functions are of the nrom $\sum_k a_k \mathbb{1}_{E_k}$ for $E_k \in \mathcal{M}$ and $\mu(E_k) < \infty$
- Let $f \geq 0$ measurable. Then there are $\varphi_k \nearrow$ simple functions such that $\varphi_k(x) \nearrow f(x)$ pointwise.
- If f is measurable, then there exist φ_k simple function such that $|\varphi_k| \nearrow$ and $\varphi_k(x) \to f(x)$ for all x
- If $f_n : E \to \mathbb{R}$ are measurable, $\mu(E) < \infty$, and $f_n \to f$ pointwise on E, then for all $\epsilon > 0$, there is an $A_\epsilon \subseteq E$ such that $\mu(E \setminus A_\epsilon)$ and $f_n \to f$ uniformly on $E \setminus A_\epsilon$
- We define $\int_X g(x)\, d\mu(x)$ for all g bounded and supported on a set of finite measure
- If $f \geq 0$, we define $\int_X f(x)\, d\mu(x) := \sup_{0 \leq g \leq f} \int_X g(x)\, d\mu(x)$ where the sup is taken over g bounded and supported on a set of finite measure.

We still have Fatou's lemma and dominated convergence. Also, we can define $L^1(X, \mu)$ and $L^2(X, \mu)$ in the analogous ways.

Lecture 24: Lebesgue-Stieltjes Integral (11/26)

Let F be increasing on $\mathbb{R}$. As we know, F has at most countably many discontinuities, and they are all jump-like. We denote

$$\lim_{\substack{x \to x_0 \\ x < x_0}} F(x) = F(x_0^-)$$
$$\lim_{\substack{x \to x_0 \\ x > x_0}} F(x) = F(x_0^+)$$

Modify F so that $F(x_0) = F(x_0^+)$, making it right continuous. Then we say that F is **normalized**.

Theorem 31. *Let F be increasing and normalized on $\mathbb{R}$. Then there is a unique measure μ (sometimes denoted dF) on the Borel sets of $\mathbb{R}$ such that*

$$\mu((a, b]) = F(b) - F(a) \qquad \text{if } a \leq b$$

Conversely, if μ is a measure on $\mathcal{B}$ that is finite on bounded intervals, then

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \\ \mu((0, -x]) & \text{if } x < 0 \end{cases}$$

Proof. Let $\mathcal{A}$ be the algebra of sets generated by intervals of type $(a, b]$. We define a premeasure $\mu_0 : \mathcal{A} \to [0, \infty]$ by $\mu_0((a, b]) = F(b) - F(a)$. This is extended to unions by forcing the additivity property. Note that here we use continuity from the right. The rest follows from Caratheodory extension. $\square$

Observation 0.10. If F is increasing and normalized, then with the μ as guaranteed from the measure, we usually denote

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \int_{\mathbb{R}} f(x)\, dF(x)$$

Observation 0.11. If F has bounded variation with values in $\mathbb{C}$, one can extend the definition naturally by applying the above process to the decomposition of F into an increasing and a decreasing part.

Observation 0.12. (Exercise) If F is absolutely continuous, then F' exists a.e and

$$\int_a^b f(x)\, dF(x) = \int_a^b f(x)F'(x)\, dx$$

Note that in particular

$$\mu((a,b]) = \int_a^b F'(x)\, dx$$

Absolute continuity of measures

Let $\mathcal{M}$ a σ-algebra of sets in X. A **signed measure** ν on $\mathcal{M}$ is a map $\nu : \mathcal{M} \to (-\infty, \infty)$ such that if $(E_j)_{j\in\mathbb{N}}$ is disjoint, then

$$\nu\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \nu(E_j)$$

note that the right series must be absolutely convergent for this to hold, since the E_j can be rename and shuffled without changing the union.

Example 0.15. For $f \in L^1$, $\nu(E) := \int_E f(x)\, dx$ gives a signed measure.

Given a signed measure ν, we define $|\nu|$, the total variation of ν, as

$$|\nu|(E) = \sup \sum_{j\in\mathbb{N}} |\nu(E_j)|$$

where the sup is taken over decompositions $\bigcup_{j\in\mathbb{N}} E_j = E$ where the E_j are disjoint.

Proposition 17. Given a signed measure ν, $|\nu|$ is a positive measure and $\nu \leq |\nu|$.

Proof. The inequality is clear. Now, let E_j disjoint in $\mathcal{M}$ and consider $E = \bigcup_j E_j$. We will show that $\sum_j |\nu|(E_j) = |\nu|(E_j)$.

Let $c_j < |\nu|(E_j)$ and chosen to be close in value for each j. By definition, for each j we can choose $E_j = \bigcup_{i\in\mathbb{N}} F_{ij}$ a disjoint union such that

$$c_j \leq \sum_i |\nu(F_{ij})|$$

Since $E = \bigcup_i \bigcup_j F_{ij}$, we have that

$$\begin{aligned}\sum_j c_j &\leq \sum_i \sum_j |\nu(F_{ij})| \\ &\leq |\nu|(E)\end{aligned}$$

so the ($\leq$) is true since this holds for any c_j.

Now, let $(F_k)_{k\in\mathbb{N}}$ be a decomposition of E. Consider $(F_k \cap E_j)_{k\in\mathbb{N}}$, which is a partition of E_j. Then we have

$$\begin{aligned}\sum_k |\nu(F_k)| &\leq \sum_k \sum_j |\nu(F_k \cap E_j)| \\ &= \sum_j \Big(\sum_k |\nu(F_k \cap E_j)|\Big) \\ &\leq \sum_j |\nu|(E_j)\end{aligned}$$

Taking a sup on the left, we are done. $\square$

Given a signed measure ν, we define

$$\begin{aligned}\nu^+ &= \frac{1}{2}(|\nu| + \nu) \\ \nu^- &= \frac{1}{2}(|\nu| - \nu)\end{aligned}$$

and note that $\nu = \nu^+ - \nu^-$ while $\nu^+ = \nu^+ + \nu^-$.

Also, we say that ν is σ-finite if $|\nu|$ is σ-finite.

Let ν and μ be two signed measures. We say that ν and μ are **mutually singular**, denoted $\nu \perp \mu$, if there are $A, B \in \mathcal{M}$ such that $A \cap B = \emptyset$ such that $\nu(E) = \nu(A \cap E)$ and $\mu(E) = \mu(E \cap B)$, meaning they are supported on disjoint sets.

We say that ν is **absolutely continuous** with respect to a measure $\mu \geq 0$, denoted $\nu \ll \mu$ if

$$\mu(E) = 0 \implies \nu(E) = 0 \quad (1)$$

Example 0.16. If ν is given by $\nu(E) = \int_E f(x)\,dx$, then $\nu \ll m$, the Lebesgue measure.

In fact, for this example, for any $\epsilon > 0$, there is some $\delta > 0$ such that $\mu(E) < \delta \implies \nu(E) < \epsilon$. We call this condition (2).

Proposition 18. (2) $\implies$ (1) and (1) $\implies$ (2) if $|\nu|$ is finite.

Proof. (2) $\implies$ (1) is clear. It suffices to prove this for $|\nu|$, so let us assume $\nu \geq 0$ for ease.

Assume (1) for ν. If (2) does not hold, then for each n there is an $E_n \in \mathcal{M}$ such that

$$\mu(E_n) < \frac{1}{2^n} \qquad \nu(E_n) \geq \epsilon$$

for some $\epsilon > 0$. Define $E^* = \limsup_n E_n = \bigcap_{n=1}^{\infty} E_n^*$ where $E_n^* = \bigcup_{k\geq n} E_k$. We know that

$$\mu(E_n^*) \leq \sum_{k\geq n} \frac{1}{2^n} < \frac{1}{2^{n-1}}$$

so $\mu(E_n^*) \to 0$, meaning $\mu(E^*) = 0$. Then we must have $\nu(E^*)$. Note that

$$\begin{aligned}
\nu(E^*) &= \lim_n \nu(E_n^*) && \text{finiteness} \\
&\geq \nu(E_n) \\
&\geq \epsilon
\end{aligned}$$

For any n. $\square$

Theorem 32 (Lebesgue, Radon, Nikodym)**.** *Let μ be a σ-finite positive measure on $(X,\mathcal{M})$, and ν a σ-finite signed measure on $\mathcal{M}$. Then there exists a unique decomposition $\nu = \nu_a + \nu_s$ of signed measures such that $\nu_a \ll \mu$, $\nu_s \perp \mu$.*

Moreover, there exists an f that is μ-integrable (in an extended sense) such that

$$\nu_a(E) = \int_E f(x)\, d\mu(x)$$

Example 0.17. Take F increasing and absolutely continuous. Then F' exists a.e, and $F(b) - F(a) = \int_a^b F'(x)\, dx$. Since the Lebesgue-Stieltjes integral satisifies $\mu \ll dx$, and

$$\mu((a,b]) = \int_a^b F'(x)\, dx$$

this decomposiiton $\mu = F'dx$ is an example of the theorem being applied.

Lecture 25: Radon-Nikodym (12/3)

Proof of Radon-Nikodym.

Case 1: Both ν and μ are finite and positive.

Denote $\rho = \nu + \mu$ and look at the linear map $l : L^2(X,\rho) \to \mathbb{C}$ given by

$$l(\varphi) = \int_X \varphi(x)\, d\nu(x)$$

We have that l is continuous, since

$$\begin{aligned}
|l(\varphi)| &\leq \int |\varphi(x)|\, d\nu(x) \\
&\leq (\nu(X))^{1/2} \|\varphi\|_{L^2(d\nu)} && \text{Cauchy-Schwarz} \\
&\leq (\nu(X))^{1/2} \|\varphi\|_{L^2(d\rho)}
\end{aligned}$$

so $\|l\| \leq (\nu(X))^{1/2} < \infty$. By the Riesz representation theorem (we prove this later), there is a $g \in L^2(X, d\rho)$ such that $l(\varphi) = \langle \varphi, \overline{g} \rangle$ where

$$\langle h_1, h_2 \rangle = \int_X h_1 \overline{h}_2\, d\rho(x)$$

This means that, for any $\varphi \in L^2(X, d\rho)$,

$$\int_X \varphi(x)\, d\nu(x) = \int_X \varphi(x) g(x)\, d\rho(x) \qquad (*)$$

Observe that if $\varphi = \mathbb{1}_E$ (say $E \in \mathcal{M}$ with $\rho(E) > 0$), then

$$\begin{aligned}\int_E d\nu &= \int_E g(x)\, d\rho(x) \\ 0 < \rho(E) &= \int_E g(x)\, d\rho(x)\end{aligned}$$

so $g > 0$ for a.e.x (with respect to ρ). Also, we know have that

$$\begin{aligned}\int_E g(x)\, d\rho(x) &= \nu(E) \\ &\leq \rho(E) \\ \frac{1}{\rho(E)} \int_E g(x)\, d\rho(x) &\leq 1\end{aligned}$$

giving that $g \leq 1$ a.e. This means that we can write $(*)$ as

$$\int_X \varphi(1-g)\, d\nu = \int_X \varphi g\, d\mu \qquad (**)$$

Consider $A = \{x \in X \mid 0 \leq g(x) < 1\}$ and $B = \{x \in X \mid g(x) = 1\}$. Define $\nu_a(E) = \nu(A \cap E)$ and $\nu_s(E) = \nu(B \cap E)$. Note that $\nu = \nu_a + \nu_s$. Also, we have that $\nu_s \perp \mu$. To see this, note that $\operatorname{supp}(\nu_s) \subseteq B$, so we will show that $\mu(B) = 0$. This follows from taking $\varphi = \mathbb{1}_B$ in $(**)$.

Now, take $\varphi = \mathbb{1}_E(1 + g + \ldots + g^n)$ in $(**)$, so that

$$\int_E (1 - g^{n+1})\, d\nu = \int_E g(1 + g + \ldots + g^n)\, d\mu$$

Note that $1 - g^{n+1} \to 0$ if $x \in B$ and $\to 1$ if $x \in A$, meaning that $1 - g^{n+1} \to \mathbb{1}_A$. Thus,

$$\int_E (1 - g^{n+1})\, d\nu \to \nu(A \cap E) = \nu_a(E) \qquad \text{dominated convergence}$$

The right side is

$$\int_E g(1 + g + \ldots + g^n)\, d\mu \to \int_E \frac{g}{1-g}\, d\mu \qquad \text{monotone convergence}$$

We define $f = \frac{g}{1-g}$. Note that $f \in L^1(X, d\mu)$ because its integral is equal to the limit of the left side $\int_E (1 - g^{n+1}) d\nu$, which is finite by assumption. Then the proof is done, modulo some minor checks.

Case 2: μ and ν are σ-finite and positive.

For this case, pick sets E_j that are disjoint such that $X = \bigcup_{j \in \mathbb{N}} E_j$ with $\mu(E_j) < \infty$ and $\nu(E_j) < \infty$ for each j. Then define measures $\mu_j(E) = \mu(E \cap E_j)$ and $\nu_j(E) = \nu(E \cap E_j)$.

Then by finiteness we can apply case 1 to get decompositions $\nu_j = \nu_{j,a} + \nu_{j,s}$. Finally, define $\nu_j = \sum_j \nu_{j,a}$ and $\mu_j = \sum_j \mu_{j,s}$.

Case 3: General case

Just split $\nu = \nu_+ - \nu_-$, apply case 2 to each, and find that everything works out.

Uniqueness: Say we have $\nu = \nu_a + \nu_s = \tilde{\nu}_a + \tilde{\nu}_s$. Then use that $\nu_a - \tilde{\nu}_a = \nu_s - \tilde{\nu}_s$. Note that absolute continuity and singularity are preserved under addition. This gives that both sides are zero. $\square$

Theorem 33 (Riesz)**.** *Let $l : \mathcal{H} \to \mathbb{C}$ be a continuous linear functional on $\mathcal{H}$ a Hilbert space. Then there exists a unique $g \in \mathcal{H}$ such that $l(f) = \langle f, g\rangle$ for any $f \in \mathcal{H}$. Moreover, $\|l\| = \|g\|_{\mathcal{H}}$.*

Proof. Note that uniqueness is clear, since if there were g_1, g_2, subtracting gives $\langle f, g_1 - g_2\rangle = 0$ for all $f \in \mathcal{H}$, so $g_1 - g_2 = 0$.

Let $l \in \mathcal{H}^*$. Consider $\ker(l) = \{f \in \mathcal{H} \mid l(f) = 0\}$. This is a closed linear subspace. Moreover, $\mathcal{H} = \ker(l) \oplus \ker(l)^\perp$. Pick $h \in \ker(l)^\perp$, with $\|h\|_2 = 1$. Define $g = \overline{l(h)}h$. Note the equivalences

$$\begin{aligned} l(f) = \langle f, g\rangle &= \langle f, \overline{l(h)}h\rangle \\ \iff \langle l(f)h, h\rangle &= \langle l(h)f, h\rangle \\ \iff \langle l(f)h - l(h)f, h\rangle &= 0 \end{aligned}$$

Note that $l(f)h - l(h)f \in \ker(l)$, so we are done. □

Hausdorff Measure

Let $E \subset \mathbb{R}^d$. Define the exterior α-dimensional Hausdorff measure of E by

$$m_\alpha^*(E) = \lim_{\delta\to 0} \inf\Big\{\sum_{k=0}^{\infty}\Big[\mathrm{diam}(F_k)\Big]^\alpha \mid E \subseteq \bigcup_{k=0}^{\infty} F_k,\ \mathrm{diam}(F_k) \le \delta\Big\}$$

This is also denoted $\lim_{\delta\to 0} H_\alpha^\delta(E)$. The diameter is defined $\mathrm{diam}(F_k) = \sup_{x,y\in F_k}|x - y|$.

Lecture 26: Hausdorff Measure (12/5)

The Hausdorff measure has nice properties (which we will not prove, due to similarity with prior cases):

- $E_1 \subseteq E_2 \implies m_\alpha^*(E_1) \le m_\alpha^*(E_2)$
- $m_\alpha^*(\bigcup_{j\in\mathbb{N}} E_j) \le \sum_{j\in\mathbb{N}} m_\alpha^*(E_j)$
- If $d(E_1, E_2) > 0$, then $m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$

Thus, m_α^* is a metric exterior measure. In particular, using a result that we have proven, m_α^* is a measure when restricted to the Borel σ-algebra. We will call the restriction m_α, which we call the α-dimensional Hausdorff measure. Other properties include:

- $m_\alpha(E + h) = m_\alpha(E)$ for all $h \in \mathbb{R}^d$
- $m_\alpha(R(E)) = m_\alpha(E)$ for any R rotation
- $m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$ for $\lambda > 0$
- $m_0(E)$ counts the number of points in E
- $m_d(E) = C_d m(E)$ for some $C_d \in \mathbb{R}$, where m is the Lebesgue measure

Proposition 19. Let $\alpha > 0$. Then

If $m_\alpha(E) < \infty$ and $\beta > \alpha$, then $m_\beta(E) = 0$

If $m_\alpha(E) > 0$ and $\beta < \alpha$, then $m_\beta(E) = \infty$

Proof. We first prove the first statement. Let E such that $m_\alpha(E) < \infty$ and choose $\beta > \alpha$. If $\operatorname{diam}(F) < \delta$, then

$$\operatorname{diam}(F)^\beta = \operatorname{diam}(F)^{\beta-\alpha}\operatorname{diam}(F)^\alpha \le \delta^{\beta-\alpha}\operatorname{diam}(F)^\alpha$$

Then, we have

$$H_\beta^\delta(E) \le \delta^{\beta-\alpha} m_\alpha^*(E)$$

Sending $\delta \to 0$ shows that $m_\beta(E) = 0$ (since $m_\alpha(E)$ is finite).

For the second statement, let E such that $m_\alpha(E) > 0$ and let $\beta < \alpha$. If $m_\beta(E) < \infty$, then $m_\alpha(E) = 0$, contradicting the first statement. Thus, $m_\beta(E) = \infty$. □

From this proposition, we know that given $E \subseteq \mathbb{R}^d$ Borel, we know that there is a unique $\alpha \in \mathbb{R}_{\ge 0}$ such that

$$m_\beta(E) = \begin{cases} \infty & \beta < \alpha \\ 0 & \beta > \alpha \end{cases}$$

This α is called the **Hausdorff dimension** of E.

Theorem 34. *The ternary Cantor set $\mathcal{C}$ has Hausdorff dimension $\alpha = \frac{\log 2}{\log 3}$. Moreover, $m_\alpha(\mathcal{C}) = 1$.*

Proof. The construction starts with 1 interval of length 1, then 2 intervals of length $\frac{1}{3}$, then 2^2 intervals of length $\frac{1}{3^2}$, and so on. Since $\mathcal{C}$ is the intersection of all of these layers, it can be covered by any of these layers. Indeed, $\mathcal{C}$ can be covered by 2^j intervals each of length $3^{-j} = \delta$. Thus,

$$\begin{aligned} H_\alpha^{3^{-j}}(\mathcal{C}) &= 2^j 3^{-\alpha j} \\ &= 2^j 3^{-\frac{\log 2}{\log 3} j} \\ &= 2^j 2^{-j} \\ &= 1 \end{aligned}$$

This gives that, $m_\alpha(\mathcal{C}) \le 1$. Now, we show that $m_\alpha(\mathcal{C}) \ge 1$. We will show something stronger, namely that for any collection of intervals $\mathcal{I}$ that cover $\mathcal{C}$, one has $\sum_{I \in \mathcal{I}} |I|^\alpha \ge 1$.

Choose such an $\mathcal{I}$, and note that we can assume it finite since $\mathcal{C}$ is compact. Moreover, we can assume all intervals are open (we can slightly extend them and take limits as necessary). We will decompose $I = J \cup J' \cup K$ for disjoint sets where $|J|, |J'| \le |K|$, so $|K| \ge \frac{|J| + |J'|}{2}$. Now, suppose further that $|I|^s \geq (|J|+|J'|+|K|)^s$ for $s \in (0,1)$. Then

$$\begin{aligned}
|I|^s &\geq (|J|+\left|J'\right|+|K|)^s \\
&\geq \left[\frac{3}{2}(|J|+\left|J'\right|)\right]^s \\
&= \frac{3^s}{2^s}(|J|+\left|J'\right|)^s \\
&= 2(|J|+\left|J'\right|)^s \\
&= 2\frac{|J|^s+|J'|^s}{2} && \text{since } s \in (0,1) \\
&= |J|^s+\left|J'\right|^s
\end{aligned}$$

Assume without loss of generality that there is a j_0 large such that any interval of size 3^{-j_0} belongs to one of the intervals of the cover (we can pick j_0 arbitrarily large so we can enlarge the intervals of our cover by arbitrarily small amounts). Then for fixed $I \in \mathcal{I}$, we claim

$$|I|^\alpha \geq \sum_{\substack{|I_{j_0}|=3^{-j_0} \\ I_{j_0} \subseteq I}} |I_{j_0}|^\alpha$$

It suffices to prove this claim.

$\square$

Ergodic Theory

Let $(X, \mathcal{M}, \mu)$ be a measure space. A map $\tau : X \to X$ is **measure-preserving** if for all $E \in \mathcal{M}$, $\tau^{-1}(E) = \{x \in X \mid \tau(x) \in E\} \in \mathcal{M}$ and $\mu(\tau^{-1}(E)) = \mu(E)$. Observe that for *tau* measure-preserving,

$$\int_X f(\tau(x))d\mu(x) = \int_X f(x)d\mu(x)$$

Example 0.18. Here are examples of measure-preserving maps:

- $X = \mathbb{Z}, \quad \tau(x) = x+1$
- $X = \mathbb{R}, \quad \tau(x) = x+h$
- $X = S^1, \quad \tau(x) =$ rotation by a fixed angle α
- $X = (0,1], \quad \tau(x) = 2x \bmod 1$

The first three are isomorphisms. The last one, however, is not injective.

The main task is to study averages of type

$$A_n f(x) = \frac{1}{n}\sum_{k=0}^{n} f(\tau^k(x))$$

Consider $L^2(X,\mu)$, and the operator $T : L^2 \to L^2$ given by $Tf(x) = f(\tau(x))$. Then we have $\|Tf\|_2 = \|f\|_2$ for any $f \in L^2$. In particular, T is an isometry. Define the set of fixed points $S = \{f \in L^2 \mid Tf = f\}$. This is a closed subspace in L^2. Then we can consider $P : L^2 \to L^2$ the orthogonal projecton onto S.

Theorem 35. *Let T be an isometry on a Hilbert space $\mathcal{H}$, and let P be the orthogonal projection onto the subspace of invariant vectors of T. Let $A_n = \frac{1}{n}(I + T + \ldots + T^{n-1})$. Then for any $f \in \mathcal{H}$,*

$$\|A_n f\| \xrightarrow{n} \|Pf\|$$

Proof. Define $S_* = \{f \in \mathcal{H} \mid T^* f = f\}$ and $S_1 = \{f \in \mathcal{H} \mid f = g - Tg, \text{ for some } g \in \mathcal{H}\} = \text{range}(I - T)$.

Lemma 12. *$S = S_*$ and $\overline{S}_1^{\perp} = S$.*

Proof of lemma. Since T is an isometry, $\langle Tf, Tg \rangle = \langle f, g \rangle$ for any $f, g \in \mathcal{H}$. Thus if $Tf = f$, we have $T^* f = T^*(Tf) = f$. For the other containment, if $T^* f = f$, then

$$\langle f, T^* f - f \rangle = 0 \implies \langle f, T^* f \rangle - \langle f, f \rangle = 0$$

Meaning that $\langle Tf, f \rangle = \|f\|^2 = \|Tf\| \|f\|$. This is the case of equality in Cauchy-Schwarz, so $Tf = cf$ for some c. In fact, $c = 1$, so $Tf = f$.

Now, let $f \in \overline{S}_1^{\perp}$. Then

$$\begin{aligned} &\langle f, (I - T)g \rangle = 0 \\ \iff &\langle (I - T)^* f, g \rangle = 0 \\ \iff &(I - T^*) f = 0 \\ \iff &T^* f = f \end{aligned}$$

giving that $f \in S_* = S$. Using the equivalences in the other direction, we get the other containment, so we are done. □

Now, $\mathcal{H} = S \oplus \overline{S}_1$. Any f splits as $f = f_0 + f_1$, where $f_0 \in S$ and $f_1 \in \overline{S}_1$. We can pick also $\tilde{f}_1 \in S_1$ such that $\left\| f_1 - \overline{f}_1 \right\| < \epsilon$ (for some fixed $\epsilon > 0$). Then we can look at

$$A_n f = A_n f_0 + A_n \tilde{f}_1 + A_n (f_1 - \tilde{f}_1)$$

We look at the summands independently

$$\begin{aligned} A_n f_0 &= \frac{1}{n} \sum_{k=0}^{n} T^k f_0 \\ &= f_0 \\ &= Pf \end{aligned}$$

$$\begin{aligned} \tilde{f}_1 &= g - Tg \text{ for some } g \\ A_n \tilde{f}_1 &= \frac{1}{n} \sum_{k=0}^{n} T^k (I - T) g \\ &= \frac{1}{n} \sum_{k} T^k g - T^{k+1} g \\ &\xrightarrow{n} 0 \end{aligned}$$

$$
\begin{aligned}
\left\|A_n(f_1-\tilde{f}_1)\right\| &= \frac{1}{n}\sum_k \left\|T^k(f_1-\tilde{f}_1)\right\| \\
&\le \epsilon
\end{aligned}
$$

Thus, taking ϵ small, we are done. $\square$

A map $\tau : X \to X$ is called **ergodic** if $f(x) = f(\tau(x))$ for all x means that $f = c$ is a constant.

Corollary 11. If τ is ergodic on a finite measure space, then

$$
\frac{1}{n}\sum_k f(\tau^k(x)) \xrightarrow{n} \int_X f \, d\mu
$$